

## UNITARIZABLE REPRESENTATIONS OF QUIVERS

THORSTEN WEIST AND KOSTYANTYN YUSENKO

ABSTRACT. We investigate representations of  $*$ -algebras associated with posets. Unitarizable representations of the corresponding (bound) quivers (which are polystable representations for some appropriately chosen slope function) give rise to representations of these algebras. Considering posets which correspond to unbound quivers this leads to an ADE-classification which describes the unitarization behaviour of their representations. Considering posets which correspond to bound quivers, it is possible to construct unitarizable representations starting with polystable representations of related unbound quivers which can be glued together with a suitable direct sum of simple representations. Finally, we estimate the number of complex parameters parametrizing irreducible unitary non-equivalent representations of the corresponding algebras.

## INTRODUCTION

In the last forty years, finite-dimensional representations of quivers (respectively posets) in the category of vector spaces, also called linear representations in the following, became a huge field of research. It turned out that one can distinguish between (unbound) quivers of representation finite type described and studied by Gabriel in [10], tame type described and studied by Nazarova in [22] and the infinite class of quivers of wild type whereas it was proved by Drozd in [8] that every representation infinite algebra is either tame or wild. Kleiner and Nazarova respectively described the posets of representation finite type and representation tame type (see [27, Theorem 10.1 and Theorem 15.3]). We refer to Section 1.1 for precise definitions of (bound) quivers, posets and for the connection between certain representations of (bound) quivers and representations of posets which are all assumed to be finite-dimensional throughout the paper.

During the last years there developed an increasing interest in a full subcategory of the category of linear representations, which is the category of representations of quivers and posets respectively in the category of Hilbert spaces, see for instance [17, 18, 20]. Namely, one can straightforwardly transfer the definitions from the linear to the Hilbert case keeping in mind that the morphisms between two representations should preserve the Hilbert structure, i.e. they are unitary maps. This restriction produces ' $*$ -wild' problems already in very simple situations (see [21]), i.e. the classification problem contains the classification of two self-adjoint matrices up to unitary isomorphism as a subproblem, see [23, Chapter 3] for more details. Hence, the idea is to consider representations which satisfy some additional conditions. That is one of the motivations for studying *orthoscalar representations* of quivers and posets, see [17, 20] and Section 1.2 for a precise definition. It makes the subject even more interesting that there are lots of topics which are closely related to such representations. For instance such representations are connected to Hermann Weyl's problem of describing the spectra of the sum of two Hermitian matrices and its generalizations, see [9] and Section 1.2 for a precise statement of the problem.

To study orthoscalar representations it is convenient to use the language of  $*$ -algebras and their  $*$ -representations (see Section 1.2 for the definition of  $*$ -algebras). With a poset  $\mathcal{N}$  consisting of  $n$  elements and a fixed tuple  $\chi = (\chi_0; \chi_1, \dots, \chi_n) \in \mathbb{R}_+^{n+1}$ , called *weight* in what follows, we

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associate the following  $*$ -algebra over the complex numbers

$$\mathcal{A}_{\mathcal{N},\chi} = \mathbb{C} \left\langle p_1, \dots, p_n \left| \begin{array}{l} p_i = p_i^* = p_i^2 \\ p_j p_i = p_i p_j = p_i, \quad i \prec j \\ \chi_1 p_1 + \dots + \chi_n p_n = \chi_0 e \end{array} \right. \right\rangle,$$

where  $e$  denotes the identity element. Two problems naturally arise: to find those weights  $\chi$  for which the algebra  $\mathcal{A}_{\mathcal{N},\chi}$  has at least one non-zero representation; for appropriate  $\chi$ , to describe all irreducible  $*$ -representations up to unitary equivalence. The second problem could turn out to be "hopeless", i.e. the isomorphism classes of  $*$ -representations can depend on arbitrarily many continuous parameters, or the algebra can be even  $*$ -wild.

In this paper we only consider finite-dimensional representations of  $\mathcal{A}_{\mathcal{N},\chi}$ . Whereas every representation of  $\mathcal{A}_{\mathcal{N},\chi}$  canonically defines a representation of  $\mathcal{N}$  in the category of vector spaces, it is not clear which kind of linear representations of  $\mathcal{N}$  allow a choice of a Hermitian metric such that they give rise to  $*$ -representations of  $\mathcal{A}_{\mathcal{N},\chi}$  (those objects that possess such a choice are called *unitarizable* following [20]). One of the essential parts of this paper is devoted to this problem. Namely, we say that a representation  $(V_0; (V_i)_{i \in \mathcal{N}})$  of  $\mathcal{N}$  can be *unitarized* with the weight  $\chi$  if it possesses a choice of a Hermitian structure in  $V_0$  in such a way that for the corresponding projections  $P_i : V_0 \rightarrow V_i$  the following equality holds

$$\chi_1 P_1 + \dots + \chi_n P_n = \chi_0 \mathbb{I}.$$

We use the fact that unitarizable representations of posets can be identified with polystable representations of quivers, i.e. representations which can be decomposed into stable ones of the same slope, going back to King's work [15], see Section 1.3 for more details. This approach turns out to be very useful because lots of statements which are known for stable quiver representations can be used and easily transferred to representations of posets.

After recalling the notion of general representations of quivers in Section 2.2 as introduced by Schofield, see [26], we state our first result, Theorem 11, saying that each general Schurian representation of a poset corresponding to an unbound quiver can be unitarized with a certain weight and hence gives rise to a  $*$ -representation of  $\mathcal{A}_{\mathcal{N},\chi}$ . Recall that Schurian representations are representations whose endomorphism ring consists only of elements of the base field. Moreover, we specify those weights  $\chi$  in terms of the dimension vector of the corresponding representation. This result is later used to show that the corresponding Hermitian operators are rigid in the sense of N.Katz in [14], see Section 3.2 for a precise definition and Theorem 27 for the statement.

In Section 2.3 we consider quivers (which are also related to posets) which are bound by some ideal as recalled in Section 1.1. Actually, this covers plenty of interesting cases of quivers which were not considered in [17, 20]. On the one hand Section 2.3 gives an explicit construction of stable representations of bound quivers including the description of the linear form defining the stability condition. On the other hand, since the constructed representations are unitarizable representations of the corresponding posets it is again possible to understand them as representations in the category of Hilbert spaces. More detailed, starting with a polystable representation of a related unbound quiver we glue its direct summands together with a direct sum of simple representations, which corresponds to the vertex where the relations determining the ideal start, in such a way that the resulting representations are stable and satisfy the relations corresponding to the ideal. This result can also be seen as a starting point of a classification of unitarizable representations of non-primitive posets, see Remark 6 for more details. Recall that a primitive poset is the cardinal sum of linearly ordered sets, see also Section 1.1 for more details.

We start Section 3 by illustrating this method on examples and finish it stating the next result which classifies posets corresponding to unbound quivers due to their unitarization behaviour. To do so we make also use of results of [12, 26]. More detailed we state that if the poset is of representation finite type, then every indecomposable representation is unitarizable, see [11], if it is primitive and of representation tame type then every Schurian representations is unitarizable,

which essentially follows from [12]. What we show is that in the representation wild case, in addition to the unitarizable representations described in Theorem 11, there are also non-unitarizable Schurian representations which depend on an arbitrary number of complex parameters. It is remarkable that the classification mentioned in the beginning of the introduction also appears in this situation.

Let us remark that it is still an open question whether or not each Schurian representation of a non-primitive tame poset is unitarizable. Nevertheless, the article gives an idea how to handle these cases. Thereby, the main idea is to extend some non-primitive subposet and its representations appropriately using the methods of Section 2.3.

The main result of the last section concerns the number of parameters that parametrize non-equivalent irreducible  $*$ -representations of  $\mathcal{A}_{\mathcal{N},\chi}$ . If  $\mathcal{N}$  is primitive and of representation finite type, then it is known that one discrete parameter parametrizes all irreducible nonequivalent representations (see [19]) and that if  $\mathcal{N}$  is primitive and of representation tame type, it is at most one continuous parameter depending on the weight  $\chi$  (see [1] and references therein). We show that if  $\mathcal{N}$  is of representation wild type (primitive or non-primitive), then there exists a weight  $\chi_{\mathcal{N}}$  such that there are families of non-equivalent irreducible  $*$ -representations of  $\mathcal{A}_{\mathcal{N},\chi_{\mathcal{N}}}$  which depend on arbitrary many continuous parameters; conjecturally such algebras are of  $*$ -wild representation type.

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#### 1. PRELIMINARIES

**1.1. Representations of quivers and posets.** Let  $Q$  be a finite quiver which is given by a set of vertices  $Q_0$  and a set of arrows  $Q_1$  denoted by  $\rho : q \rightarrow q'$  for  $q, q' \in Q_0$ . The vertex  $q$  is called *tail*, and the vertex  $q'$  is called *head* of the arrow  $\rho$ . A vertex  $q \in Q_0$  is called *sink* if there does not exist an arrow  $\rho : q \rightarrow q' \in Q_1$ . A vertex  $q \in Q_0$  is called *source* if there does not exist an arrow  $\rho : q' \rightarrow q \in Q_1$ .

For a vertex  $q \in Q_0$  let

$$N_q = \{q' \in Q_0 \mid \exists \rho : q \rightarrow q' \vee \exists \rho : q' \rightarrow q\}$$

be the set of its neighbours.

In the following we only consider finite quivers without oriented cycles. Define the abelian group

$$\mathbb{Z}Q_0 = \bigoplus_{q \in Q_0} \mathbb{Z}q$$

and the monoid of dimension vectors  $\mathbb{N}Q_0$ .

Let  $k$  be an algebraically closed field. A finite-dimensional  $k$ -representation of  $Q$  is given by a tuple

$$X = ((X_q)_{q \in Q_0}, (X_\rho)_{\rho \in Q_1} : X_q \rightarrow X_{q'})$$

of finite-dimensional  $k$ -vector spaces and  $k$ -linear maps between them. A morphism of representations  $f : X \rightarrow Y$  is a tuple  $f = (f_q : X_q \rightarrow Y_q)_{q \in Q_0}$  of  $k$ -linear maps such that  $Y_\rho f_q = f_{q'} X_\rho$  for

all  $\rho : q \rightarrow q'$ . We denote by  $\text{Rep}(Q)$  the abelian category of finite-dimensional representations of  $Q$ .

We call a representation  $X$  *Schurian* if  $\text{End}(X) := \text{Hom}(X, X) = k$ . We say that  $X$  is *strict* if all maps  $X_\rho$  are injective. The dimension vector  $\underline{\dim} X \in \mathbb{N}Q_0$  of  $X$  is defined by

$$\underline{\dim} X = \sum_{q \in Q_0} \dim X_q \cdot q.$$

Let  $\alpha \in \mathbb{N}Q_0$  be a dimension vector. The variety  $R_\alpha(Q)$  of  $k$ -representations of  $Q$  with dimension vector  $\alpha$  is defined as the affine  $k$ -space

$$R_\alpha(Q) = \bigoplus_{\rho: q \rightarrow q'} \text{Hom}(k^{\alpha_q}, k^{\alpha_{q'}}).$$

The algebraic group  $G_\alpha = \prod_{q \in Q_0} \text{GL}_{\alpha_q}(k)$  acts on  $R_\alpha(Q)$  via simultaneous base change, i.e.

$$(g_q)_{q \in Q_0} * (X_\rho)_{\rho \in Q_1} = (g_{q'} X_\rho g_q^{-1})_{\rho: q \rightarrow q'}.$$

The orbits are in bijection with the isomorphism classes of  $k$ -representations of  $Q$  with dimension vector  $\alpha$ .

Let  $kQ$  be the path algebra of  $Q$  and let  $RQ$  be the arrow ideal, see [2, Chapter II.1] for a definition. A relation in  $Q$  is a  $k$ -linear combination of paths of length at least two which have the same head and tail. For a set of relations  $(r_j)_{j \in J}$  we can consider the admissible ideal  $I$  generated by these relations, where admissible means that we have  $RQ^m \subseteq I \subseteq RQ^2$  for some  $m \geq 2$ . Now a representation  $X$  of  $Q$  is bound by  $I$ , and thus a representation of the bound quiver  $(Q, I)$ , if  $X_{r_j} = 0$  for all  $j \in J$ . For every dimension vector this defines a closed subvariety of  $R_\alpha(Q)$  denoted by  $R_\alpha(Q, I)$ . If  $R$  is a minimal set of relations generating  $I$ , by  $r(q, q', I)$  we denote the number of relations with starting vertex  $q$  and terminating vertex  $q'$ . Following [4], for the dimension of  $R_\alpha(Q, I)$  we get

$$\dim R_\alpha(Q, I) \geq \dim R_\alpha(Q) - \sum_{(q, q') \in (Q_0)^2} r(q, q', I) \alpha_q \alpha_{q'}.$$

Let  $C_{(Q, I)}$  be the Cartan matrix of  $(Q, I)$ , i.e.  $c_{q', q} = \dim e_q(kQ/I)e_{q'}$  where  $e_q$  denotes the primitive idempotent (resp. the trivial path) corresponding to the vertex  $q$ . On  $\mathbb{Z}Q_0$  a non-symmetric bilinear form, the Euler characteristic, is defined by

$$\langle \alpha, \beta \rangle := \alpha^T (C_{(Q, I)}^{-1})^T \beta.$$

Then for two representation  $X$  and  $Y$  we have

$$\langle X, Y \rangle := \langle \underline{\dim} X, \underline{\dim} Y \rangle = \sum_{i=0}^{\infty} (-1)^i \dim \text{Ext}^i(X, Y),$$

see for instance [2, Proposition 3.13]. Moreover, if  $Q$  is unbound, for two representations  $X, Y$  of  $Q$  with  $\underline{\dim} X = \alpha$  and  $\underline{\dim} Y = \beta$  we have

$$\langle X, Y \rangle = \dim \text{Hom}(X, Y) - \dim \text{Ext}(X, Y) = \sum_{q \in Q_0} \alpha_q \beta_q - \sum_{\rho: q \rightarrow q' \in Q_1} \alpha_q \beta_{q'}$$

and  $\text{Ext}^i(X, Y) = 0$  for  $i \geq 2$ , see [25, Section 2].

As usual we call a dimension vector  $\alpha \in \mathbb{N}Q_0$  a root of the quiver  $Q$  if there exists an indecomposable representation of  $Q$  with  $\underline{\dim} X = \alpha$ . A root is called real if  $\langle \alpha, \alpha \rangle = 1$  and imaginary otherwise.

Let  $X$  and  $Y$  be two representations of a quiver  $Q$ . Then we can consider the linear map

$$\gamma_{X, Y} : \bigoplus_{q \in Q_0} \text{Hom}(X_q, Y_q) \rightarrow \bigoplus_{\rho: q \rightarrow q' \in Q_1} \text{Hom}(X_q, Y_{q'})$$

with  $\gamma_{X,Y}((f_q)_{q \in Q_0}) = (Y_\rho f_q - f_{q'} X_\rho)_{\rho: q \rightarrow q' \in Q_1}$ .

We have  $\ker(\gamma_{X,Y}) = \text{Hom}(X, Y)$  and  $\text{coker}(\gamma_{X,Y}) = \text{Ext}(X, Y)$ , see [25, Section 2]. The first statement is obvious. The second one follows because every exact sequence  $E(f) \in \text{Ext}(X, Y)$  is defined by a morphism  $f \in \bigoplus_{\rho: q \rightarrow q' \in Q_1} \text{Hom}(X_q, Y_{q'})$  in the following way

$$0 \rightarrow Y \rightarrow ((Y_q \oplus X_q)_{q \in Q_0}, \left( \begin{pmatrix} Y_\rho & f_\rho \\ 0 & X_\rho \end{pmatrix} \right)_{\rho \in Q_1}) \rightarrow X \rightarrow 0$$

with the canonical inclusion on the left hand side and the canonical projection on the right hand side. Now it is straightforward to check that two sequences  $E(f)$  and  $E(g)$  are equivalent if and only if  $f - g \in \text{Im}(\gamma_{X,Y})$ .

As far as bound quivers are concerned, we just have to consider those exact sequences such that the middle term also satisfies the relations, thus we have  $\text{Ext}_{(Q,I)}(X, Y) \subseteq \text{Ext}_Q(X, Y)$ .

Let  $\mathcal{N}$  be a finite poset where  $\prec$  denotes the partial order in  $\mathcal{N}$ . A finite-dimensional representation of  $\mathcal{N}$  is given by a collection of finite-dimensional  $k$ -vector spaces

$$V = (V_0; (V_q)_{q \in \mathcal{N}})$$

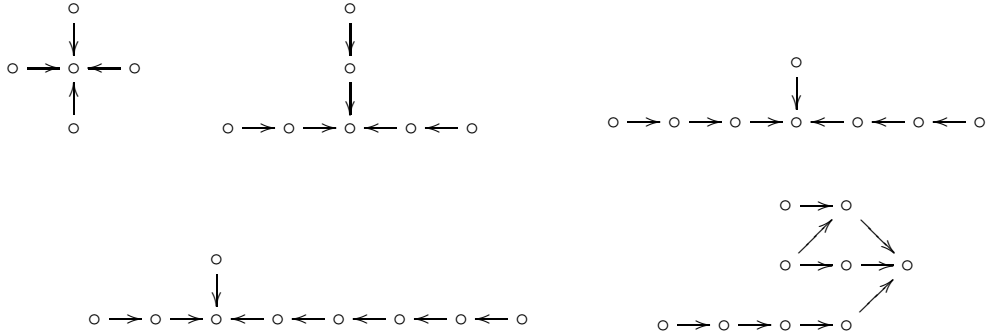
such that  $V_q \subseteq V_0$  for all  $q \in \mathcal{N}$  and  $V_q \subseteq V_{q'}$  if  $q \prec q'$ . A morphism between two representations  $(V_0; (V_q)_{q \in \mathcal{N}})$  and  $(W_0; (W_q)_{q \in \mathcal{N}})$  is given by a  $k$ -linear map  $g: V_0 \rightarrow W_0$  such that  $g(V_q) \subseteq W_q$  for all  $q \in \mathcal{N}$ . By  $\text{Rep}(\mathcal{N})$  we denote the additive category of representations of a poset  $\mathcal{N}$ .

Denote by  $\mathcal{N}^0$  the extension of  $\mathcal{N}$  by a unique maximal element  $q_0$ . With  $\mathcal{N}$  we associate the Hasse quiver of  $\mathcal{N}^0$  which will be denoted by  $Q(\mathcal{N})$ , i.e. we orient all edges of the Hasse diagram of the poset  $\mathcal{N}$  to the unique maximal element. The other way around let  $Q$  be a connected quiver without oriented cycles and multiple arrows. Moreover, we assume that all arrows are oriented to one vertex  $q_0$  which is called the root. To  $Q$  we can naturally associate the poset  $\mathcal{N}(Q) = Q_0 \setminus \{q_0\}$  such that  $q \prec q'$  if and only if there exists a path from  $q$  to  $q'$ . A poset is said to be *primitive* if it is the disjoint (cardinal) sum of linearly ordered sets  $L_i$  of order  $n_i$ . In this case we denote the poset and corresponding quiver by  $(n_1, \dots, n_s)$ . Since the quiver  $Q(\mathcal{N})$  corresponding to a primitive poset  $\mathcal{N}$  has a star-shaped form, it is called *star-shaped*.

We recall the classification of posets by their representation type (see for example [27, Chapter 10 and Chapter 15] for precise definitions of the terms finite, tame and wild representation type).

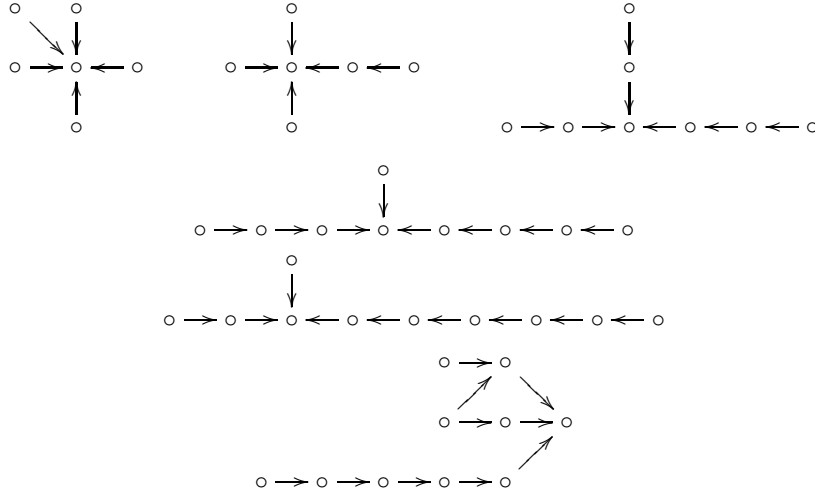
**Theorem 1.** (Kleiner [27, Theorem 10.1] and Nazarova [27, Theorem 15.3])

- (1) A poset  $\mathcal{N}$  is of representation finite type if and only if the quiver  $Q(\mathcal{N})$  does not contain any of the following critical quivers



as a proper subquiver.

- (2) A poset  $\mathcal{N}$  is of representation tame type if and only if the quiver  $Q(\mathcal{N})$  does not contain any of the following critical quivers



as a proper subquiver.

In the following, the two non-primitive posets in the previous theorem will be denoted by  $(N, 4)$  and  $(N, 5)$  respectively.

We briefly recall the relation between representations of posets and representations of bound quivers. Everything presented here is well-known, see for instance [7] for a more general setup. Let  $Q(\mathcal{N})$  be the quiver induced by a poset  $\mathcal{N}$ . Let  $\alpha \in \mathbb{N}Q(\mathcal{N})_0$  be a dimension vector. By  $S_\alpha(Q(\mathcal{N})) \subset R_\alpha(Q(\mathcal{N}))$  we denote the (possibly empty) open subvariety of strict representations. For every (non-oriented) cycle  $\rho_1 \dots \rho_n \tau_m^{-1} \dots \tau_1^{-1}$  with  $\rho_i, \tau_j \in Q(\mathcal{N})_1$  and  $\rho_i \neq \tau_j$  we define a relation

$$\rho_1 \dots \rho_n - \tau_1 \dots \tau_m.$$

Let  $I$  be the ideal generated by all such relations.

Let  $V = (V_0; (V_q)_{q \in \mathcal{N}})$  be a representation of  $\mathcal{N}$  with dimension vector  $\alpha$ . This defines a representation  $F(V) \in S_\alpha(Q(\mathcal{N}), I)$  satisfying the stated relations. Indeed, every inclusion  $V_q \subset V_{q'}$  defines an injective map  $F(V)_{\rho_{q,q'}} : V_q \rightarrow V_{q'}$ . For two arbitrary representations  $V$  and  $W$  a morphism  $g : V \rightarrow W$ , defines a morphism  $F(g) : F(V) \rightarrow F(W)$  where  $F(g)_q := g|_{V_q} : F(V)_q \rightarrow F(W)_q$ .

The other way around let  $X \in S_\alpha(Q(\mathcal{N}), I)$ . This gives rise to a representation  $G(X)$  of  $\mathcal{N}$  by defining  $G(X)_q = X_{\rho_n^q} \circ \dots \circ X_{\rho_1^q}(X_{q_0})$  for some path  $\rho_1^q \dots \rho_n^q$  from  $q$  to  $q_0$ . This definition is independent of the chosen path. Moreover, every morphism  $f = (f_q)_{q \in Q_0} : X \rightarrow Y$  defines a morphism  $G(f)$  which is induced by  $f_{q_0} : X_{q_0} \rightarrow Y_{q_0}$ .

Thus we get an equivalence between the categories of strict representations of  $Q(\mathcal{N})$  bound by  $I$  and representations of  $\mathcal{N}$ . This equivalence also preserves dimension vectors.

If the global dimension of  $kQ(\mathcal{N})/I$  is at most two, see for instance [2, Chapter A.4] for a definition, for two representations  $X$  and  $Y$  with  $\underline{\dim} X = \alpha$  and  $\underline{\dim} Y = \beta$  we get

$$\begin{aligned} \langle X, Y \rangle &= \dim \operatorname{Hom}(X, Y) - \dim \operatorname{Ext}^1(X, Y) + \dim \operatorname{Ext}^2(X, Y) \\ &= \sum_{q \in Q(\mathcal{N})_0} \alpha_q \beta_q - \sum_{\rho: q \rightarrow q' \in Q(\mathcal{N})_1} \alpha_q \beta_{q'} + \sum_{(q, q') \in (Q(\mathcal{N})_0)^2} r(q, q', I) \alpha_q \beta_{q'}, \end{aligned}$$

see [4]. This defines a quadratic form  $q_{Q(\mathcal{N})}(\alpha) := \langle \alpha, \alpha \rangle$ , often called Tits form (in some cases it coincides with the Drozd form for posets as introduced in [7], see also [28] for the connection between different quadratic forms associated with posets).

In order to shorten notation and because we are also only interested in bound representation of  $Q(\mathcal{N})$  if it has unoriented cycles, we denote by  $Q(\mathcal{N})$  the quiver  $Q(\mathcal{N})$  bound by  $I$  as constructed above. In particular, if  $\mathcal{N}$  contains pairwise disjoint elements  $q_1, q_2, q_3$  such that  $q_1 \prec q_2$  and  $q_1 \prec q_3$ , the quiver  $Q(\mathcal{N})$  is bound, otherwise it is unbound.

**1.2. Orthoscalar representations of posets and quivers. \*-Algebras associated to posets and graphs.** Let  $k = \mathbb{C}$ . Fix a poset  $\mathcal{N}$  and the corresponding quiver  $Q(\mathcal{N})$ . Let  $\mathcal{H}$  be the category of (finite-dimensional) Hilbert spaces. Consider the subcategory  $\text{Rep}(Q(\mathcal{N}), \mathcal{H})$  of  $\text{Rep}(Q(\mathcal{N}))$  consisting of representations  $X$  such that  $X_q$  are Hilbert spaces for all  $q \in Q_0$ . Denoting by  $X_\rho^*$  the adjoint linear map of  $X_\rho$  for a morphism  $f : X \rightarrow Y$  we additionally require that  $f_q Y_\rho^* = X_\rho^* f_{q'}$  for all  $\rho : q \rightarrow q'$ . It is straightforward to check (see for example [23, Chapter 1] for a similar statement) that two representations  $X$  and  $Y$  are isomorphic in  $\text{Rep}(Q(\mathcal{N}), \mathcal{H})$ , i.e. there exists an invertible morphism  $f : X \rightarrow Y$ , if and only if they are unitary isomorphic, i.e.  $f_\rho$  is a unitary linear map for every  $\rho \in Q(\mathcal{N})_1$ . Now we may easily transfer this definition to  $\text{Rep}(\mathcal{N})$  and form the category  $\text{Rep}(\mathcal{N}, \mathcal{H})$ . Even in simple cases the description of indecomposable objects in the category  $\text{Rep}(\mathcal{N}, \mathcal{H})$  is a very hard, so-called \*-wild, problem [21]. Thus it is natural to consider subcategories of  $\text{Rep}(\mathcal{N}, \mathcal{H})$ .

We say that an object  $V = (V_0, (V_q)_{q \in \mathcal{N}})$  of  $\text{Rep}(\mathcal{N}, \mathcal{H})$  is *orthoscalar* if there exists a *weight*  $\chi = (\chi_0, (\chi_q)_{q \in \mathcal{N}}) \in \mathbb{R}^{|\mathcal{N}|+1}$  such that

$$(1.1) \quad \sum_{q \in \mathcal{N}} \chi_q P_q = \chi_0 P_0$$

where the  $P_i$  is the orthogonal projection onto the subspace  $V_i$ . Denote this category, which is a full subcategory of  $\text{Rep}(\mathcal{N}, \mathcal{H})$ , by  $\text{Rep}(\mathcal{N}, \mathcal{H})_{\text{os}}$ . We should mention that the term *locally-scalar* instead of orthoscalar also appears in the literature (see [20]). Note that if  $\alpha \in \mathbb{N}^{|\mathcal{N}|+1}$  is the dimension vector of  $V$  by taking the trace on both sides of (1.1) we obtain

$$\sum_{q \in \mathcal{N}} \chi_q \alpha_q = \chi_0 \alpha_0.$$

Every object of  $\text{Rep}(\mathcal{N}, \mathcal{H})_{\text{os}}$  can be identified with an object of  $\text{Rep}(\mathcal{N})$  applying the forgetful functor. Thus it is natural to ask for which kind of representations  $V = (V_0, (V_q)_{q \in \mathcal{N}})$  we can choose a hermitian form such that there exists a weight  $\chi = (\chi_0, (\chi_q)_{q \in \mathcal{N}}) \in \mathbb{R}^{|\mathcal{N}|+1}$  such that the corresponding orthoprojections  $P_q$  onto subspaces  $V_q$  satisfy (1.1). In this case, following [20, Section 4] we say that  $V$  is *unitarizable* with (or can be *unitarized* with) the weight  $\chi$ .

Recall that a *\*-algebra* is an algebra  $\mathcal{A}$  over  $\mathbb{C}$  together with an anti-automorphism  $*$  :  $\mathcal{A} \rightarrow \mathcal{A}$ , i.e.  $*$  is an algebra automorphism such that  $(ab)^* = b^* a^*$  and  $(a^*)^* = a$  for all  $a, b \in \mathcal{A}$ .

Assume that  $\mathcal{N}$  consists of  $n$  points. For a given weight  $\chi = (\chi_0; \chi_1, \dots, \chi_n) \in \mathbb{R}_+^{n+1}$  consider the \*-algebra defined by

$$\mathcal{A}_{\mathcal{N}, \chi} = \mathbb{C} \left\langle p_1, \dots, p_n \left| \begin{array}{l} p_i = p_i^* = p_i^2 \\ \chi_1 p_1 + \dots + \chi_n p_n = \chi_0 e \\ p_j p_i = p_i p_j = p_i, \quad i \prec j \end{array} \right. \right\rangle$$

where  $e$  denotes the identity element of  $\mathcal{A}_{\mathcal{N}, \chi}$ .

The objects of  $\text{Rep}(\mathcal{N}, \mathcal{H})_{\text{os}}$  correspond to finite-dimensional \*-representations of  $\mathcal{A}_{\mathcal{N}, \chi}$  and results about the structure of these objects can be formulated in terms of representations of  $\mathcal{A}_{\mathcal{N}, \chi}$ . Let us describe how these algebras are related to \*-algebras associated with star-shaped graphs (considered for example in [1, 19]) in the case when the poset is primitive.

Let  $\Gamma = (\Gamma_0, \Gamma_1)$  be a connected graph with vertices  $\Gamma_0$  and edges  $\Gamma_1$ . For a given poset  $\mathcal{N}$  by  $\Gamma(\mathcal{N})$  we denote the underlying graph of the quiver  $Q(\mathcal{N})$ . We call  $\Gamma$  *star-shaped* if it is the underlying graph of a star-shaped quiver. Clearly  $\Gamma(\mathcal{N})$  is star-shaped if and only if  $\mathcal{N}$  is primitive.

Assuming that the graph  $\Gamma$  is of the type  $(m_1, \dots, m_n)$  we identify the set of vertices  $\Gamma_0$  with  $(g_0; g_i^{(j)})$ , where  $g_0$  is the root vertex and  $g_{i_1}^{(j)}$  and  $g_{i_2}^{(j)}$  lie on the same branch of  $\Gamma$ . Fixing some vector  $\omega = (\omega_0; \omega_i^{(j)}) \in \mathbb{R}_+^{|\Gamma_0|}$  with  $\omega_{i_1}^{(j)} > \omega_{i_2}^{(j)}$  if  $i_1 > i_2$  (following [1, 19] a vector with such

properties is called *character*), we consider the  $*$ -algebra

$$\mathcal{B}_{\Gamma, \omega} = \mathbb{C} \left\langle a_1, \dots, a_n \mid \begin{array}{l} a_i = a_i^* \\ (a_i - \omega_1^{(i)}) \dots (a_i - \omega_{m_i}^{(i)}) = 0 \\ a_1 + \dots + a_n = \omega_0 e \end{array} \right\rangle.$$

Any  $*$ -representation of  $\mathcal{B}_{\Gamma, \omega}$  in some Hilbert space is given by an  $n$ -tuple of Hermitian operators with spectra  $\sigma(A_i) \in \{\omega_1^{(i)} < \dots < \omega_{m_i}^{(i)}\}$  such that

$$A_1 + \dots + A_n = \omega_0 \mathbb{I}.$$

Recall that the last equation is connected with generalizations of Hermann Weyl's problem: can one describe the eigenvalues of the sum of two Hermitian  $n \times n$ -matrices in terms of the eigenvalues of the two single matrices, see also [9] for the description of the classical problem of Hermann Weyl and generalizations. Note that Klyachko, see [16], solved a more general version of this problem. The interested reader should also consult [1, 19] and references therein for generalizations. Fixing a finite-dimensional representation of  $\mathcal{B}_{\Gamma, \omega}$  in some Hilbert space  $H$ , for each operator  $A_i$  we can consider its spectral decomposition

$$A_i = \omega_1^{(i)} \tilde{P}_1^{(i)} + \dots + \omega_{m_i}^{(i)} \tilde{P}_{m_i}^{(i)}.$$

If the poset  $\mathcal{N}$  is primitive of type  $(m_1, \dots, m_n)$ , then each  $*$ -representation of  $\mathcal{A}_{\mathcal{N}, \chi}$  generates a  $*$ -representation of  $\mathcal{B}_{\Gamma(\mathcal{N}), \omega}$  for some character  $\omega$  which can be written in terms of the weight  $\chi$ . More precisely, let  $(P_i^{(j)})$  be a  $*$ -representation of  $\mathcal{A}_{\mathcal{N}, \chi}$ , which means that  $P_{i_1}^{(j)} P_{i_2}^{(j)} = P_{i_2}^{(j)} P_{i_1}^{(j)} = P_{i_1}^{(j)}$  if  $i_1 < i_2$  and

$$\chi_1^{(1)} P_1^{(1)} + \dots + \chi_{m_1}^{(1)} P_{m_1}^{(1)} + \dots + \chi_1^{(n)} P_1^{(n)} + \dots + \chi_{m_n}^{(n)} P_{m_n}^{(n)} = \chi_0 \mathbb{I}.$$

Letting  $\tilde{P}_1^{(j)} = P_1^{(j)}$ ,  $\tilde{P}_i^{(j)} = P_i^{(j)} - P_{i-1}^{(j)}$  and taking the weight  $\omega_{m_j}^{(j)} = \chi_{m_j}^{(j)}$ ,  $\omega_i^{(j)} = \chi_i^{(j)} + \omega_{i+1}^{(j)}$ ,  $\omega_0 = \chi_0$ , we get a representation of  $\mathcal{B}_{\Gamma(\mathcal{N}), \omega}$ . Note that one can prove that  $\mathcal{A}_{\mathcal{N}, \chi}$  and  $\mathcal{B}_{\Gamma(\mathcal{N}), \omega}$  are isomorphic using the same transformation between the projections.

**1.3. Stable representations and unitarizable representations of quivers.** In order to study representations of the algebra  $\mathcal{A}_{\mathcal{N}, \chi}$ , we are going to use the notion of stable quiver representations. In the space of  $\mathbb{Z}$ -linear functions  $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}Q_0, \mathbb{Z})$  we consider the basis given by the elements  $q^*$  for  $q \in Q_0$ , i.e.  $q^*(q') = \delta_{q, q'}$  for  $q \in Q_0$ . Define  $\dim := \sum_{q \in Q_0} q^*$ . After choosing  $\Theta \in \text{Hom}_{\mathbb{Z}}(\mathbb{Z}Q_0, \mathbb{Z})$ , we define the slope function  $\mu : \mathbb{N}Q_0 \setminus \{0\} \rightarrow \mathbb{Q}$  via

$$\mu(\alpha) = \frac{\Theta(\alpha)}{\dim(\alpha)}.$$

The slope  $\mu(\underline{\dim} X)$  of a representation  $X$  of  $Q$  is abbreviated to  $\mu(X)$ .

**Definition 2.** A representation  $X$  of  $(Q, I)$  is semistable (resp. stable) if for all proper subrepresentations  $0 \neq U \subsetneq X$  the following holds:

$$\mu(U) \leq \mu(X) \text{ (resp. } \mu(U) < \mu(X)).$$

Denote the set of semistable (resp. stable) points by  $R_{\alpha}^{ss}(Q, I)$  (resp.  $R_{\alpha}^s(Q, I)$ ).

It is well-known that the definition of  $\mu$ -stability is equivalent to that of A. King in [15]. Let  $\tilde{\Theta}$  be another linear form. A representation  $X$  such that  $\tilde{\Theta}(\underline{\dim} X) = 0$  is semistable (resp. stable) in the sense of King if and only if

$$\tilde{\Theta}(\underline{\dim} U) \geq 0 \text{ (resp. } \tilde{\Theta}(\underline{\dim} U) > 0)$$

for all subrepresentations  $U \subset X$  (resp. all proper subrepresentations  $0 \neq U \subsetneq X$ ).

In this situation we have the following theorem summarising several main results of [15]:

**Theorem 3.** (1) The set of stable points  $R_{\alpha}^s(Q, I)$  is an open subset of the set of semistable points  $R_{\alpha}^{ss}(Q, I)$ , which is an open subset of  $R_{\alpha}(Q, I)$ .



- (2) *There exists a categorical quotient  $M_\alpha^{ss}(Q, I) := R_\alpha^{ss}(Q, I)/G_\alpha$ . Moreover,  $M_\alpha^{ss}(Q, I)$  is a projective variety.*
- (3) *There exists a geometric quotient  $M_\alpha^s(Q, I) := R_\alpha^s(Q, I)/G_\alpha$ , which is a smooth subvariety of  $M_\alpha^{ss}(Q, I)$ .*

**Remark 1.**

- The moduli space  $M_\alpha^{ss}(Q, I)$  does not parametrize the semistable representations, but the polystable ones. Polystable representations are such representations which can be decomposed into stable ones of the same slope, see also [15].
- For a stable representation  $X$  we have that its orbit is of maximal possible dimension, see [15]. Since the scalar matrices act trivially on  $R_\alpha(Q, I)$ , the isotropy group is one-dimensional. Thus, if the moduli space is not empty, for the dimension of the moduli space we have the lower bound

$$\begin{aligned} \dim M_\alpha^s(Q, I) &= \dim R_\alpha(Q, I) - (\dim G_\alpha - 1) \\ &\geq 1 - \sum_{q \in Q} \alpha_q^2 + \sum_{\rho: q \rightarrow q' \in Q_1} \alpha_q \alpha_{q'} - \sum_{(q, q') \in Q_0 \times Q_0} r(q, q', I) \alpha_q \alpha_{q'}. \end{aligned}$$

Moreover, if  $I = 0$  and the moduli space is not empty, we have

$$\dim M_\alpha^s(Q) = 1 - \langle \alpha, \alpha \rangle.$$

Finally, we point out some properties of (semi-)stable representations. These properties will be very useful at different points of this paper, for proofs see for instance [24, Section 4].

**Lemma 4.** *For a bound quiver  $(Q, I)$  let  $0 \rightarrow Y \rightarrow X \rightarrow Z \rightarrow 0$  be a short exact sequence of representations.*

- (1) *The following are equivalent:*
- (a)  $\mu(Y) \leq \mu(X)$
  - (b)  $\mu(X) \leq \mu(Z)$
  - (c)  $\mu(Y) \leq \mu(Z)$
- The same holds when replacing  $\leq$  by  $<$ .*
- (2) *The following holds:  $\min(\mu(Y), \mu(Z)) \leq \mu(X) \leq \max(\mu(Y), \mu(Z))$ .*
- (3) *If  $\mu(Y) = \mu(X) = \mu(Z)$ , then  $X$  is semistable if and only if  $Y$  and  $Z$  are semistable.*
- (4) *Every stable representation is Schurian.*

If some property is independent of the point chosen in some non-empty open subset  $\mathcal{O}$  of  $R_\alpha(Q)$ , following [26], we say that this property is true for a general representation with dimension vector  $\alpha \in \mathbb{N}Q_0$ .

Denote by  $\beta \hookrightarrow \alpha$ , if a general representation of dimension  $\alpha$  has a subrepresentation of dimension  $\beta$ . A root  $\alpha$  is called Schur root if there exists a representation  $X$  with  $\underline{\dim} X = \alpha$  such that  $\text{End}(X) = k$ . By [26, Section 1] it follows that in this case there already exists an open subset of Schurian representations. From [26, Theorem 6.1] we get the following theorem:

**Theorem 5.** *Let  $\alpha$  be a dimension vector of the quiver  $Q$ . Then  $\alpha$  is a Schur root if and only if for all  $\beta \hookrightarrow \alpha$  we have  $\langle \beta, \alpha \rangle - \langle \alpha, \beta \rangle > 0$ .*

Thus, if we define  $\Theta_\alpha := \langle \_, \alpha \rangle - \langle \alpha, \_ \rangle$ , a general representation of dimension  $\alpha$  is  $\Theta_\alpha$ -stable in the sense of King if and only if  $\alpha$  is a Schur root.

Consider the  $n$ -subspace quiver  $S(n)$ , i.e.  $S(n)_0 = \{q_0, q_1, \dots, q_n\}$  and  $S(n)_1 = \{\rho_i : q_i \rightarrow q_0 \mid i \in \{1, \dots, n\}\}$ . Define the slope  $\mu$  by choosing  $\Theta = (-1, 0, \dots, 0)$ . Then we have the following lemma:

**Lemma 6.** *A representation  $X$  of  $S(n)$  with dimension vector  $\alpha$  is  $\mu$ -stable if and only if  $X$  is  $\Theta_\alpha$ -stable.*

*Proof.* Let  $U$  be a subrepresentation of dimension  $\beta$ . It is easy to check that we have

$$\frac{-\alpha_{q_0}}{\sum_{i=0}^n \alpha_{q_i}} > \frac{-\beta_{q_0}}{\sum_{i=0}^n \beta_{q_i}}$$

if and only if

$$\langle \beta, \alpha \rangle - \langle \alpha, \beta \rangle = \sum_{i=1}^n \alpha_{q_i} \beta_{q_0} - \sum_{i=1}^n \beta_{q_i} \alpha_{q_0} > 0.$$

□

**Remark 2.**

- Note that in general, it is not possible to choose a slope function  $\mu$  once and for all such that general representations of all Schur roots are  $\mu$ -stable. Actually, this is only the case if the rank of the anti-symmetrized adjacency matrix of the quiver has rank equal to two, see [29].

We will need the following lemma:

**Lemma 7.** *Let  $Y$  and  $Z$  be two representations of a bound quiver  $(Q, I)$  such that  $\text{Hom}(Y, Z) = \text{Hom}(Z, Y) = 0$  and  $\text{End}(Z) = k$ . Let  $\dim \text{Ext}(Z, Y) = d_0 > 0$ . Let  $e_1, \dots, e_d \in \text{Ext}(Z, Y)$  with  $1 \leq d \leq d_0$  be linear independent. Consider the exact sequence*

$$e : 0 \rightarrow Y \rightarrow X \rightarrow Z^d \rightarrow 0$$

*induced by  $e_1, \dots, e_d$ . Then we have  $\text{End}(X) \subseteq \text{End}(Y)$ .*

*Proof.* Consider the following long exact sequence

$$0 \longrightarrow \text{Hom}(Z, Y) = 0 \longrightarrow \text{Hom}(Z, X) \longrightarrow \text{Hom}(Z, Z^d) \xrightarrow{\phi} \text{Ext}(Z, Y)$$

induced by  $e$ , see [2, Section A.4] for more details. By construction  $\phi$  is injective and, therefore,  $\text{Hom}(Z, X) = 0$ . Now consider the following commutative diagram induced by  $e$ :

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Hom}(Z^d, Y) = 0 & \longrightarrow & \text{Hom}(Z^d, X) & \longrightarrow & \text{Hom}(Z^d, Z^d) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Hom}(X, Y) & \longrightarrow & \text{Hom}(X, X) & \longrightarrow & \text{Hom}(X, Z^d) \\ & & \downarrow & & \downarrow \phi_1 & & \downarrow \\ 0 & \longrightarrow & \text{Hom}(Y, Y) & \xrightarrow{\phi_2} & \text{Hom}(Y, X) & \longrightarrow & \text{Hom}(Y, Z^d) = 0 \end{array}$$

Now we also have  $\text{Hom}(Z^d, X) = 0$ . Thus,  $\phi_1$  is also injective and since  $\phi_2$  is an isomorphism, the claim follows. □

Note that the dual lemma dealing with sequences of the form

$$0 \rightarrow Z^d \rightarrow X \rightarrow Y \rightarrow 0$$

also holds and can be proven analogously.

## 2. UNITARIZATION

**2.1. Criteria for being unitarizable.** Except for Section 2.3, in the following, we fix the base field  $\mathbb{C}$ . Recall that we may understand a strict representation  $X$  of a (bound) quiver  $Q(\mathcal{N})$  associated to a poset  $\mathcal{N}$  as a system of vector subspaces  $(V_0; (V_q)_{q \in \mathcal{N}})$  and vice versa. We will use the following criteria for  $\chi$ -unitarization of  $X$  (which was basically obtained by the different authors A. King [15], B. Totaro [30], A. Klyachko [16], Y. Hu [13] and others in different formulations).

**Theorem 8.** *Let  $(V_0; (V_q)_{q \in \mathcal{N}})$  be an indecomposable strict representation. Then  $(V_0; (V_q)_{q \in \mathcal{N}})$  is unitarizable with the weight  $\chi = (\chi_0; (\chi_q)_{q \in \mathcal{N}}) \in \mathbb{R}_+^{|\mathcal{N}|+1}$  if and only if for every proper subspace  $0 \neq U \subsetneq V_0$  the following holds*

$$\chi_0 = \frac{1}{\dim V_0} \sum_{q \in \mathcal{N}} \chi_q \dim V_q,$$

$$\frac{1}{\dim U} \sum_{q \in \mathcal{N}} \chi_q \dim(V_q \cap U) < \frac{1}{\dim V_0} \sum_{q \in \mathcal{N}} \chi_q \dim V_q.$$

**Remark 3.**

- If an indecomposable representation  $V$  of a poset  $\mathcal{N}$  can be unitarized with the weight  $\chi \in \mathbb{N}^{|\mathcal{N}|+1}$ , the corresponding quiver representation  $X = F(V)$  is obviously  $\tilde{\Theta}$ -stable in the sense of King with

$$\tilde{\Theta} = (\chi_0, (-\chi_q)_{q \in \mathcal{N}})$$

and vice versa. Moreover, choosing the linear form  $\Theta = \mu(X) \dim -\tilde{\Theta}$ , where  $\mu(X) \in \mathbb{Z}$  can be chosen arbitrarily, this representation is  $\mu$ -stable. Moreover, we have

$$\chi_0 = \mu(X) - \Theta_0, \quad \chi_q = \Theta_q - \mu(X), \quad q \in \mathcal{N}.$$

- It is easy to check that we can modify the linear form  $\Theta$  which defines the slope  $\mu$  without changing the set of stable points in the following two ways: first we can multiply it by a positive integer; second, we can add an integer multiple of the linear form  $\dim$  to  $\Theta$ . In particular, if we change the linear form appropriately, the weight, which it defines, can be assumed to be positive.

We will also use the following lemma:

**Lemma 9.** *Let  $V = (V_0; V_1, \dots, V_n)$  be an indecomposable  $\chi$ -unitarizable representation. Then for an arbitrary set of subspaces  $V_{n+j} \subset V$ ,  $j = 1, \dots, m$ , the representation  $\tilde{V} = (V_0; V_1, \dots, V_n, V_{n+1}, \dots, V_{n+m})$  is also indecomposable and unitarizable with some weight.*

*Proof.* We prove that  $(V_0; V_1, \dots, V_n, V_{n+1})$  is unitarizable with some weight (the remaining part follows by induction). Let  $U \subset V_0$  be some subspace of  $V_0$  such that

$$R = \frac{1}{\dim V_0} \sum_{i=1}^n \chi_i \dim V_i - \frac{1}{\dim U} \sum_{i=1}^n \chi_i \dim(V_i \cap U)$$

is minimal. Note that, it is clear that such a subspace exists because the right hand side only takes finitely many values. Since  $V$  is unitarizable and indecomposable, we have  $R > 0$  and there exists an  $\varepsilon > 0$  such that  $R - \varepsilon > 0$ . Define  $\tilde{\chi}$  in the following way

$$\tilde{\chi}_i = \chi_i, \quad i = 1, \dots, n, \quad \tilde{\chi}_{n+1} = R - \varepsilon.$$

Our claim is that  $\tilde{V}$  is  $\tilde{\chi}$ -unitarizable. Indeed, let  $M \subset V_0$  be some proper subspace of  $V_0$  then we have

$$\begin{aligned} \frac{1}{\dim M} \sum_{i=1}^{n+1} \tilde{\chi}_i \dim(V_i \cap M) &= \frac{1}{\dim M} \sum_{i=1}^n \chi_i \dim(V_i \cap M) + \frac{\tilde{\chi}_{n+1} \dim(V_{n+1} \cap M)}{\dim M} \\ &\leq \frac{1}{\dim V_0} \sum_{i=1}^n \chi_i \dim V_i - R + \frac{(R - \varepsilon) \dim(V_{n+1} \cap M)}{\dim M} \\ &< \frac{1}{\dim V_0} \sum_{i=1}^n \chi_i \dim V_i < \frac{1}{\dim V_0} \sum_{i=1}^{n+1} \tilde{\chi}_i \dim V_i. \end{aligned}$$

Hence  $(V_0; V_1, \dots, V_n, V_{n+1})$  is  $\tilde{\chi}$ -unitarizable.  $\square$

**2.2. Unitarization of general representations of unbound quivers.** In this subsection we restrict to posets  $\mathcal{N}$  such that the induced quiver  $Q(\mathcal{N})$  is unbound, i.e. for the ideal of relations  $I$  we have  $I = 0$ . In particular,  $Q(\mathcal{N})$  has no oriented and unoriented cycles. Let  $\alpha \in \mathbb{N}Q(\mathcal{N})_0$  be a dimension vector. We define  $\varphi_q : N_q \rightarrow \{\pm 1\}$  by

$$\varphi_q(q') = \begin{cases} -1 & \text{if } \rho : q' \rightarrow q, \\ 1 & \text{if } \rho : q \rightarrow q'. \end{cases}$$

and the weight  $\chi(\alpha)$  by

$$(\chi(\alpha))_q = \begin{cases} \sum_{q' \in N_q} \varphi_q(q') \alpha_{q'}, & q \neq q_0 \\ -\sum_{q' \in N_q} \varphi_q(q') \alpha_{q'}, & q = q_0 \end{cases}.$$

This defines a weight function  $\chi : \mathbb{N}Q(\mathcal{N})_0 \rightarrow \mathbb{Z}Q(\mathcal{N})_0$ .

**Definition 10.** Let  $\chi : \mathbb{N}Q(\mathcal{N})_0 \rightarrow \mathbb{Z}Q(\mathcal{N})_0$  be a weight function and  $\alpha \in \mathbb{N}Q(\mathcal{N})_0$ . If we have  $(\chi(\alpha))_q \geq 0$  for every  $q \in Q(\mathcal{N})_0$ , the dimension vector  $\alpha$  is called  $\chi$ -positive.

Note that, if the poset  $\mathcal{N}$  is primitive, then each strict dimension vector is  $\chi$ -positive.

**Theorem 11.** (1) Let  $\alpha$  be a  $\chi$ -positive Schur root of the unbound quiver  $Q(\mathcal{N})$  induced by a poset  $\mathcal{N}$ . Then a general representation of  $Q(\mathcal{N})$  with dimension vector  $\alpha$  can be unitarized with the weight  $\chi(\alpha)$ .  
 (2) Let  $\alpha$  be a Schur root of the unbound quiver  $Q(\mathcal{N})$  induced by a poset  $\mathcal{N}$ . Then a general representation of  $Q(\mathcal{N})$  with dimension vector  $\alpha$  can be unitarized with a weight  $\chi'$  which is obtained by modifying  $\chi(\alpha)$  as in Remark 3.

*Proof.* By  $q_0$  we denote the unique sink. Let  $\alpha$  be a Schur root. Let  $X$  be a general representation with dimension vector  $\alpha$  and let  $\beta$  be the dimension vector of a subrepresentation of  $X$ . By Theorem 5 we have

$$\Theta_\alpha(\beta) = \langle \beta, \alpha \rangle - \langle \alpha, \beta \rangle > 0.$$

Then it is easy to check that we have

$$\begin{aligned} \langle \beta, \alpha \rangle - \langle \alpha, \beta \rangle &= - \sum_{q \in Q(\mathcal{N})_0} \beta_q \sum_{\substack{q' \in N_q \\ \varphi_q(q')=1}} \alpha'_q + \sum_{q \in Q(\mathcal{N})_0} \beta_q \sum_{\substack{q' \in N_q \\ \varphi_q(q')=-1}} \alpha'_q \\ &= - \sum_{q \in Q(\mathcal{N})_0} \beta_q \sum_{q' \in N_q} \varphi_q(q') \alpha_{q'}. \end{aligned}$$

Recall that  $\chi_q(\alpha) = \sum_{q' \in N_q} \varphi_q(q') \alpha_{q'}$ .

By Theorem 8, a representation can be unitarized with the weight  $\chi(\alpha)$  if and only if

$$\begin{aligned} \frac{1}{\beta_{q_0}} \sum_{q \in Q(\mathcal{N})_0 \setminus \{q_0\}} \beta_q \sum_{q' \in N_q} \varphi_q(q') \alpha_{q'} &< \frac{1}{\alpha_{q_0}} \sum_{q \in Q(\mathcal{N})_0 \setminus \{q_0\}} \alpha_q \sum_{q' \in N_q} \varphi_q(q') \alpha_{q'} \\ &= \frac{1}{\alpha_{q_0}} \sum_{q \in N_{q_0}} \varphi_q(q_0) \alpha_{q_0} \alpha_q = - \sum_{q \in N_{q_0}} \varphi_{q_0}(q) \alpha_q \end{aligned}$$

for all subrepresentations  $U$  of dimension vector  $\beta$ . But this is obviously the same.

Taking into account the second part of Remark 3, the second part of the Theorem follows when changing the linear form  $\Theta_\alpha$  appropriately.  $\square$

**Corollary 12.** *Let the  $Q(\mathcal{N})$  induced by the poset  $\mathcal{N}$  be unbound. Then the unique indecomposable representation of a real root  $\alpha$  can be unitarized if and only if  $\alpha$  is a real Schur root.*

*Proof.* If  $\alpha$  is not a Schur root, we have  $\dim \text{End} X_\alpha > 1$  for the unique indecomposable representation with dimension vector  $\alpha$ . In particular,  $X_\alpha$  cannot be stable, and thus cannot be unitarized.

If  $\alpha$  is a real Schur root, the orbit of  $X_\alpha$  is dense in the affine variety  $R_\alpha(Q)$ . Indeed, as already mentioned in Section 1.3, in this case a general representation has trivial endomorphism ring and is, therefore, isomorphic to  $X_\alpha$ . Thus we can apply the preceding theorem.  $\square$

**2.3. Unitarization of general representations of bound quivers.** Let  $k$  be an algebraically closed field. In this section we state a recipe which can be used to construct stable representations of bound quivers (which are unitarizable for  $k = \mathbb{C}$ ). Let  $\mathcal{N}$  be a poset and  $Q(\mathcal{N})$  be the corresponding (bound) quiver as defined in Section 1.1. Note that, in general, we do not have  $\text{Ext}_{Q(\mathcal{N})}^i(X, Y) = 0$  if  $i \geq 2$  for two arbitrary representations  $X$  and  $Y$  of the quiver  $Q(\mathcal{N})$ . Thus, in order to obtain some result similar to Theorem 11, the basic idea is the following: we glue polystable representations of an unbound quiver, which is a subquiver of  $Q(\mathcal{N})$ , with a direct sum of a simple module in order to obtain stable representations of  $Q(\mathcal{N})$ . Note that the global dimension of the corresponding path algebra of the unbound quiver is one.

As already mentioned, we say that a general representation of dimension  $\alpha$  satisfies some property if there exists an open subset  $\mathcal{O}$  of  $R_\alpha(Q)$  such that every representation  $X_u \in \mathcal{O}$ , satisfies this property. In abuse of notation, we will skip the  $u$  in what follows. Moreover, if there is more than one property requested, we always consider elements lying in the intersection of the corresponding open subsets. In addition, when considering general representations, we restrict to dimension vectors whose support can be understood as a quiver without relations. Recall that otherwise the variety of representations can be reducible, see [15]. Consider

$$\nu : R_\alpha(Q) \times R_\beta(Q) \rightarrow \mathbb{Z}, \quad (X, Y) \mapsto \dim \text{Ext}(X, Y).$$

This function is upper semi-continuous, see for instance [26]. By  $\text{ext}(\alpha, \beta)$  we denote the minimal value of  $\nu$ .

In order to prove the main result of this section, we will frequently make use of the following result [26, Theorem 3.3]:

**Theorem 13.** *A general representation of dimension  $\alpha$  has a subrepresentation of dimension  $\beta$  if and only if  $\text{ext}(\beta, \alpha - \beta) = 0$ .*

Thus fixing a dimension vector  $\alpha$ , we can choose a general representation  $X$  in such a way that for every dimension vector  $\beta \hookrightarrow \alpha$  there exists a subrepresentation  $Y$  of dimension  $\beta$  such that  $\text{Ext}(Y, X/Y) = 0$ . Actually, in order to test a representation of dimension  $\alpha$  for stability, it is sufficient to consider one subrepresentation for any dimension vector  $\beta$  with  $\beta \hookrightarrow \alpha$  because the

slope only depends on the dimension vector.

Let  $\mathcal{N}$  be a poset corresponding to an unbound quiver and  $\mathcal{M} \subset \mathcal{N}$  be a subset of elements such that

$$t(\mathcal{M}) := \min\{q \in \mathcal{N}^0 \mid q' \preceq q \forall q' \in \mathcal{M}\}$$

is unique. If, in addition,  $\mathcal{M}$  is such that for any two elements  $q, q' \in \mathcal{M}$  we have  $t(\{q, q'\}) = t(\mathcal{M})$  we say that  $\mathcal{M}$  is *appropriate*.

**Lemma 14.** *Let  $\mathcal{M} \subset \mathcal{N}$  be an appropriate subset of  $\mathcal{N}$ . Then for a general representation  $X$  of  $Q(\mathcal{N})$  we have*

$$\dim \bigcap_{q \in \mathcal{M}} X_q = \max\{0, \sum_{q \in \mathcal{M}} \dim X_q - (|\mathcal{M}| - 1) \dim X_{t(\mathcal{M})}\}.$$

*Proof.* Let  $U$  and  $X$  be two  $k$ -vector spaces such that  $U \subsetneq X$  and let  $x \in X$ . If  $(b_1, \dots, b_{\dim U})$  is a basis of  $U$  then  $x \in U$  is equivalent to  $\text{rank}(b_1, \dots, b_{\dim U}, x) = \dim U$ . Thus  $x \in U$  is a closed condition because it is equivalent to the vanishing of all  $(\dim U + 1)$ -minors of the defined matrix.

Without loss of generality, we can assume that  $\mathcal{M} \cup t(\mathcal{M})$  corresponds to the subspace quiver  $S(n)$  for some  $n \in \mathbb{N}$ . We proceed by induction on  $n$  and on the dimension of  $X_{q_n}$  where we use the notation of Section 1.3. Assume that  $0 \leq \dim X_{q_n} < \dim X_{q_0}$ . If  $U := \bigcap_{i=1}^{n-1} X_{q_i} + X_{q_n} \neq X_{q_0}$ , let  $x \in X_{q_0}$  such that  $x \notin U$ . Let  $(b_1, \dots, b_{\dim X_{q_n}})$  be a basis of  $X_{q_n}$  and define  $\tilde{X}_{q_n} := \langle b_1, \dots, b_{\dim X_{q_n}}, x \rangle$ .

If  $U = X_{q_0}$ , let  $x \notin X_{q_n}$  and define  $\tilde{X}_{q_n}$  as before. In both cases we have

$$\begin{aligned} \dim \bigcap_{i=1}^{n-1} X_{q_i} \cap \tilde{X}_{q_n} &= \dim \bigcap_{i=1}^{n-1} X_{q_i} + \dim \tilde{X}_{q_n} - \dim(\tilde{X}_{q_n} + \bigcap_{i=1}^{n-1} X_{q_i}) \\ &= \max\{0, \sum_{i=1}^{n-1} \dim X_{q_i} + \dim \tilde{X}_{q_n} - (n-1) \dim X_{q_0}\}. \end{aligned}$$

Thus the claim follows by the first part of the proof.  $\square$

Let  $\mathcal{N}$  be a poset and let

$$\mathcal{P} = \{q \in \mathcal{N} \mid \exists q_1, q_2 \in \mathcal{N}, q_1, q_2 \text{ are incomparable}, q \prec q_1, q_2\}.$$

The poset  $\mathcal{N}' = \mathcal{N} \setminus \mathcal{P}$  is associated to an unbound quiver. We call the tuple of posets  $(\mathcal{N}', \mathcal{N})$  (resp. the tuple of corresponding quivers) related. For instance, starting with the non-primitive poset  $(N, 5)$ , we get the related primitive poset  $(2, 1, 5)$ , see Section 1 for the notation.

In the following, we assume that  $\mathcal{N}'$  and  $\mathcal{N} = \mathcal{N}' \cup \{q\}$  are related. Moreover, we assume that  $N_q$  is an appropriate subset of  $\mathcal{N}$ . The first assumption is no restriction because we will see that the case  $\mathcal{N} = \mathcal{N}' \cup \{q_1, \dots, q_n\}$  can be treated by applying Lemma 9.

Using the notation of Section 1.3 it is easy to check that we have  $r(q, t(N_q), I) = |N_q| - 1$  and  $r(l, l', I) = 0$  otherwise where  $I$  is the ideal generated by the commutativity relations as described in Section 1.1. Fixing a dimension vector, for a representation of the poset  $\mathcal{N}$  satisfying the dimension formula of Lemma 14, it is often straightforward to write down a projective and injective resolution of minimal length, see [2, Chapter I.5] for more details. Moreover, in these cases the global dimension is at most two because projective resolutions of minimal length of the simple modules  $S_q$ ,  $q \in Q(\mathcal{N})_0$ , defined by  $(S_q)_q = k$  and  $(S_q)_{q'} = 0$  if  $q' \neq q$ , have at most length two, see [2, Theorem A.4.8].

Obviously, every representation of  $Q(\mathcal{N}')$  can be naturally understood as a representation of  $Q(\mathcal{N})$ . Let  $\alpha'$  be a dimension vector of  $Q(\mathcal{N}')$  such that a general representation is polystable with respect to the linear form  $\Theta_{\alpha'}$ , i.e. the canonical decomposition of  $\alpha'$  only consists of Schur roots of the same slope, see [26] for the general theory concerning canonical decomposition and

[6] for a very useful algorithm determining the canonical decomposition. Note that in this case we have  $\langle \beta, \alpha' \rangle - \langle \alpha', \beta \rangle = 0$  for all roots  $\beta$  contained in the canonical decomposition.

Clearly, every representation of  $Q(\mathcal{N}')$  satisfies the commutativity relations of  $Q(\mathcal{N})$ . In particular, the varieties of representations corresponding to dimension vectors  $\alpha \in \mathbb{N}Q(\mathcal{N})_0$  with  $\alpha_q = 0$  are irreducible, see [15]. Let  $X' = \bigoplus_{i=1}^m (X'_i)^{t_i}$  with  $\dim X' = \alpha'$  and  $X'_i \not\cong X'_j$  for  $i \neq j$  be a general polystable representation of  $Q(\mathcal{N})$  and  $S_q$  be the simple module corresponding to  $q$ . Since we have  $\dim X'_q = 0$ , it is straightforward that we have

$$\dim \operatorname{Ext}_{Q(\mathcal{N})}(S_q, X') = \dim \bigcap_{l \in N_q} X'_l = \max\{0, \sum_{l \in N_q} \dim X'_l - (|N_q| - 1) \dim X'_{t(N_q)}\}$$

where  $t(N_q)$  is the vertex of the quiver  $Q(\mathcal{N})$  where the relations starting at  $q$  terminate. Thus, if  $\dim \bigcap_{l \in N_q} X'_l \neq \{0\}$ , we have

$$-\langle S_q, X' \rangle = \sum_{l \in N_q} \dim X'_l - (|N_q| - 1) \dim X'_{t(N_q)} = \dim \operatorname{Ext}_{Q(\mathcal{N})}(S_q, X'),$$

and, therefore, we generally have  $\operatorname{Ext}^2(S_q, X') = 0$ . Moreover, for two representations  $X'$  and  $Y'$  of  $Q(\mathcal{N}')$  we obviously have  $\operatorname{Ext}_{Q(\mathcal{N})}(X', Y') = \operatorname{Ext}_{Q(\mathcal{N}')} (X', Y')$  and  $\operatorname{Hom}_{Q(\mathcal{N})}(X', Y') = \operatorname{Hom}_{Q(\mathcal{N}')} (X', Y')$ . In the following, we will skip the index  $Q(\mathcal{N})$ , and we will only use indices if we consider the quiver  $Q(\mathcal{N}')$ .

**Definition 15.** We call a dimension vector  $\alpha'$  of  $Q(\mathcal{N}')$  strongly strict if for a general representation  $X'$  with  $\underline{\dim} X' = \alpha'$ , we have  $\operatorname{Ext}(S_q, X') \neq 0$ .

For instance, in the case of the poset  $(N, 5)$  we may consider the related poset  $(2, 1, 5)$  and the unique imaginary Schur root  $\alpha' = (6; 2, 4; 3; 1, 2, 3, 4, 5)$ . This root is strongly strict and we get a representation of the poset  $(N, 5)$  with dimension vector  $\alpha = (6; 2, 4; 1, 3; 1, 2, 3, 4, 5)$  by an extension with the simple module corresponding to the additional source.

Define  $n_i := \dim \operatorname{Ext}(S_q, X'_i)$ . Consider the quiver  $\tilde{Q}$  with vertices  $\tilde{Q}_0 = \{l_0, l_1, \dots, l_m\}$  and arrows  $\tilde{Q}_1 = \{\rho_{i,j} : l_0 \rightarrow l_j \mid i \in \{1, \dots, m\}, j \in \{1, \dots, n_i\}\}$ . Then every representation of this quiver with dimension vector  $t = (t_0, t_1, \dots, t_m)$  induces an exact sequence  $e \in \operatorname{Ext}(S_q^{t_0}, X')$  and vice versa. More detailed, keeping in mind the description of exact sequences given in Section 1.1, every exact sequence  $e$  is uniquely determined by a linear map

$$f(e) : (S_q^{t_0})_q \rightarrow \bigoplus_{i=1}^m (\cap_{q' \in N_q} (X'_i)_{q'})^{t_i},$$

i.e. a linear map  $f(e) : k^{t_0} \rightarrow \bigoplus_{i=1}^m k^{t_i n_i}$ . In turn, the components of this map define linear maps

$$X(e)_{i,j} = (f(e)_{i,1,j}, \dots, f(e)_{i,t_i,j}) : k^{t_0} \rightarrow k^{t_i}$$

for  $i = 1, \dots, m$  and  $j = 1, \dots, n_i$ , and, therefore, a representation of  $\tilde{Q}$  of dimension  $(t_0, t_1, \dots, t_m)$ . Reversing this construction, every representation of  $\tilde{Q}$  defines an exact sequence  $e \in \operatorname{Ext}(S_q^{t_0}, X')$ .

**Lemma 16.** The middle terms of  $e$  and  $e'$  are isomorphic if and only if  $X(e)$  and  $X(e')$  are isomorphic.

*Proof.* Let  $g * X(e) = X(e')$  with  $g = (g_0, g_1, \dots, g_m) \in \prod_{i=0}^m \operatorname{Gl}_{t_i}(k)$ . Since  $\operatorname{End}(X'_i) = k$ , this induces bijective endomorphisms  $g_0 : (S_q)^{t_0} \rightarrow (S_q)^{t_0}$  and  $g_i : (X'_i)^{t_i} \rightarrow (X'_i)^{t_i}$  for  $i = 1, \dots, m$ . In particular, we get the following commutative diagram

$$\begin{array}{ccc} (S_q^{t_0})_q & \xrightarrow{g_0} & (S_q^{t_0})_q \\ \downarrow (X(e)_{i,j})_{j=1, \dots, n_i} & & \downarrow (X(e')_{i,j})_{j=1, \dots, n_i} \\ (\cap_{q' \in N_q} (X'_i)_{q'})^{t_i} & \xrightarrow{g_i|_{\cap_{q' \in N_q} (X'_i)_{q'}}} & (\cap_{q' \in N_q} (X'_i)_{q'})^{t_i} \end{array}$$

showing that the middle terms associated with  $X(e)$  and  $X(e')$  are isomorphic. The other way around, assume that the middle terms of  $e$  and  $e'$  are isomorphic. Since  $\text{Hom}(S_q, X') = \text{Hom}(X', S_q) = 0$  and by the universal property of the kernel and cokernel, we naturally obtain a commutative diagram

$$\begin{array}{ccccccc} e : 0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & S_q^{t_0} \longrightarrow 0 \\ & & \downarrow g' & & \downarrow \cong & & \downarrow g_0 \\ e' : 0 & \longrightarrow & X' & \longrightarrow & \tilde{X} & \longrightarrow & S_q^{t_0} \longrightarrow 0 \end{array}$$

where  $g'$  and  $g_0$  are isomorphisms inducing an isomorphism  $(g_0, \dots, g_m) \in \prod_{i=0}^m \text{Gl}_{t_i}(k)$  between  $X(e)$  and  $X(e')$ .  $\square$

We have the following result, see [25, Theorem 1.2] where it is used that the given Schur roots (including the simple one) are pairwise orthogonal, i.e. there exist no homomorphisms between them:

**Theorem 17.** *The category of representations of  $\tilde{Q}$  is equivalent to the category of representations  $X$  of  $Q(\mathcal{N})$  having a filtration*

$$0 \rightarrow \bigoplus_{i=1}^m (X'_i)^{t_i} \rightarrow X \rightarrow S_q^{t_0} \rightarrow 0$$

for some  $t \in \mathbb{N}\tilde{Q}_0$ .

We obtain the following corollary:

**Corollary 18.** *Let  $X'$  be a polystable representation of  $Q(\mathcal{N}')$ . Then there exists a strict indecomposable (resp. Schurian) representation of  $Q(\mathcal{N})$  satisfying the commutativity relations if  $(t_0, t_1, \dots, t_n)$  is a root (resp. Schur root) of  $\tilde{Q}$ .*

We call the dimension vector  $t$  stable if it is a Schur root and polystable if it has the canonical decomposition  $t = \bigoplus_{i=1}^m \alpha_{l_i}^{t_i}$  with  $\alpha_{l_i} = l_0 + n_i l_i$ . Moreover, we call the extension  $e$  stable (resp. polystable) if the corresponding representation of  $\tilde{Q}$  is stable (resp. polystable).

We will need the following lemma:

**Lemma 19.** *Let  $X' = \bigoplus_{i=1}^m (X'_i)^{t_i}$  with  $X'_i \not\cong X'_j$  for  $i \neq j$  be a polystable representation. If  $Y'$  is an indecomposable subrepresentation of  $X'$  such that  $\text{Hom}(X', Y') \neq 0$ , it follows that  $Y' \cong X'_i$  for some  $i \in \{1, \dots, m\}$ .*

*Proof.* Let  $\Psi : X' \rightarrow Y'$  be non-zero. Since the canonical composition  $\tau : X'_j \hookrightarrow X' \xrightarrow{\Psi} Y'$  is not zero for some  $j$ , this defines a factor representation  $\text{Im}(\tau) = U$  of  $X'_j$  which is a subrepresentation of  $X'$ . Thus we have  $\mu(X'_j) \leq \mu(U) \leq \mu(X')$  and thus  $\mu(X'_j) = \mu(U) = \mu(X')$ . It follows that  $U \cong X'_j$ .

Since  $Y'$  is a subrepresentation of  $X'$ , the canonical composition  $\phi : Y' \hookrightarrow X' \cong \bigoplus_{i=1}^m (X'_i)^{t_i} \rightarrow X'_i$  defines a non-zero homomorphism  $\phi \circ \tau : X'_j \rightarrow X'_i$  for some  $i$ . It follows that  $i = j$ . Moreover, since  $\text{End}(X'_i) = k$ , we obtain that  $\phi \circ \tau$  is forced to be an isomorphism and, therefore,  $X'_j$  is a direct summand of  $Y'$ . Since  $Y'$  is indecomposable, we have  $Y' \cong X'_j$ .  $\square$

In this setup, let  $X$  be a stable extension of some general representation  $X'$  with  $\underline{\dim} X' = \alpha'$  where  $\alpha'$  is strongly strict. The remaining part of this section is dedicated to proving that we can construct stable (resp. unitarizable) representations of  $Q(\mathcal{N})$  in this way, see Theorem 24 for the precise statement.



Every subrepresentation  $Y$  of  $X$  induces a subrepresentation  $Y'$  of  $X'$ . In particular, we get a commutative diagram

$$(2.1) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & S_q^{r_0} & \longrightarrow & S_q^{t_0} & \longrightarrow & S_q^{t_0-r_0} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & X/Y \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & X'/Y' \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & & 0 & & 0 \end{array}$$

Since  $\alpha'$  is strongly strict, we have  $\text{Ext}^2(Y, X) \cong \text{Ext}^2(Y', X) = 0$ . Moreover, since  $\text{Hom}(Y', S_q) = \text{Ext}(Y', S_q) = 0$ , we get  $\text{Hom}(Y', X') \cong \text{Hom}(Y', X)$  and  $\text{Ext}(Y', X') \cong \text{Ext}(Y', X)$ . We also have  $\text{Hom}(S_q, X) = 0$ . Since we have  $\text{Ext}^2(S_q, X') = \text{Ext}^2(S_q, X) = 0$ , from the long exact sequence

$$0 \rightarrow \text{Hom}(Y, X) \rightarrow \text{Hom}(Y', X) \rightarrow \text{Ext}(S_q^{r_0}, X) \rightarrow \text{Ext}(Y, X) \rightarrow \text{Ext}(Y', X) \rightarrow 0$$

we get

$$\dim \text{Hom}(Y, X) - \dim \text{Ext}(Y, X) = \dim \text{Hom}(Y', X') - \dim \text{Ext}(Y', X') - \dim \text{Ext}(S_q^{r_0}, X).$$

Moreover, we have  $\text{Ext}^2(X, Y) \cong \text{Ext}^2(X, Y')$  and we get long exact sequences

$$0 \rightarrow \text{Hom}(X, Y') \rightarrow \text{Hom}(X, Y) \rightarrow \text{Hom}(X, S_q^{r_0}) \rightarrow \text{Ext}(X, Y') \rightarrow \text{Ext}(X, Y) \rightarrow 0$$

and

$$\begin{aligned} 0 \rightarrow \text{Hom}(X, Y') &\rightarrow \text{Hom}(X', Y') \rightarrow \text{Ext}(S_q^{t_0}, Y') \rightarrow \text{Ext}(X, Y') \\ &\rightarrow \text{Ext}(X', Y') \rightarrow \text{Ext}^2(S_q^{t_0}, Y') \rightarrow \text{Ext}^2(X, Y') \rightarrow 0 \end{aligned}$$

where  $\text{Ext}^2(X', Y') = 0$ . Thus we get

$$\begin{aligned} \langle X, Y \rangle &= \dim \text{Hom}(X', Y') - \dim \text{Ext}(X', Y') + \dim \text{Hom}(X, S_q^{r_0}) - \dim \text{Ext}(S_q^{t_0}, Y') \\ &\quad + \dim \text{Ext}^2(S_q^{t_0}, Y'). \end{aligned}$$

Thus in summary we get

$$\begin{aligned} \langle Y, X \rangle - \langle X, Y \rangle &= \langle Y', X' \rangle - \langle X', Y' \rangle - \dim \text{Ext}(S_q^{r_0}, X) - \dim \text{Hom}(X, S_q^{r_0}) \\ &\quad + \dim \text{Ext}(S_q^{t_0}, Y') - \dim \text{Ext}^2(S_q^{t_0}, Y'). \end{aligned}$$

For a strongly strict dimension vector  $\alpha$  of  $Q(\mathcal{N})$  we fix the linear form  $\Theta_\alpha : \mathbb{Z}Q_0 \rightarrow \mathbb{Z}$  given by

$$\begin{aligned} \Theta_\alpha(\beta) &= \langle \beta, \alpha \rangle - \langle \alpha, \beta \rangle \\ &= - \sum_{\rho: l \rightarrow l' \in Q(\mathcal{N})_1} \beta_l \alpha_{l'} + \sum_{(l, l') \in (Q_0)^2} r(l, l', I) \beta_l \alpha_{l'} + \sum_{\rho: l \rightarrow l' \in Q(\mathcal{N})_1} \alpha_l \beta_{l'} - \sum_{(l, l') \in (Q_0)^2} r(l, l', I) \alpha_l \beta_{l'} \\ &= - \sum_{\rho: l \rightarrow l' \in Q(\mathcal{N})_1} \beta_l \alpha_{l'} + (|N_q| - 1) \beta_q \alpha_{t(N_q)} + \sum_{\rho: l \rightarrow l' \in Q(\mathcal{N})_1} \alpha_l \beta_{l'} - (|N_q| - 1) \alpha_q \beta_{t(N_q)}. \end{aligned}$$

**Remark 4.**

- Let  $\mathcal{N} = \mathcal{N}' \cup \{q\}$  be a poset related to a poset  $\mathcal{N}'$ . Recall that  $\mathcal{N}'$  is associated with an unbound quiver. Let  $\alpha$  and  $\beta$  be two dimension vectors of  $\mathcal{N}$  and let  $\alpha'$  and  $\beta'$  be

the corresponding dimension vectors of  $\mathcal{N}'$  such that  $\beta' \hookrightarrow \alpha'$ . Consider the linear form induced by the considerations from above:

$$\begin{aligned}\tilde{\Theta}_\alpha(\beta) &= \langle \beta', \alpha' \rangle - \langle \alpha', \beta' \rangle - \beta_q \left( \sum_{l \in N_q} \alpha_l - (|N_q| - 1) \alpha_{t(N_q)} \right) + \\ &\quad \alpha_q \left( \sum_{l \in N_q} \beta_l - (|N_q| - 1) \beta_{t(N_q)} \right).\end{aligned}$$

It is straightforward that we have  $\Theta_\alpha = \tilde{\Theta}_\alpha$ .

This linear form corresponds to the following weight as we will see in the next theorem: let  $\chi'$  be the weight as given in Theorem 11. Let  $\chi$  be the weight such  $\chi_q = \sum_{l \in N_q} \alpha_l - (|N_q| - 1) \alpha_{t(N_q)}$ ,  $\chi_l = \chi'_l - \alpha_q$  for all  $l \in N_q$ ,  $\chi_{t(N_q)} = \chi'_{t(N_q)} + \alpha_q(|N_q| - 1)$  and  $\chi_l = \chi'_l$  for the remaining vertices.

Using the notation of the preceding remark we get the following statement:

**Theorem 20.** *Let  $\mathcal{N} = \mathcal{N}' \cup \{q\}$  be a poset related to a poset  $\mathcal{N}'$ . A representation  $X$  of dimension  $\alpha$  can be unitarized with the weight  $\chi$  if and only if we have  $\tilde{\Theta}_\alpha(\dim Y) > 0$  for every subrepresentation  $Y$  of  $X$ . If there is at least one such representation, there exists an open, not necessarily dense, subset of unitarizable representations.*

*Proof.* The first part follows analogously to the proof of Theorem 11. The second part follows by [26, Section 1].  $\square$

First we will show that every subrepresentation of  $X'$  does not contradict the stability condition.

**Proposition 21.** *Let  $\alpha$  be  $\chi$ -positive where the weight is given as in Remark 4. If  $r_0 = 0$ , i.e.  $Y = Y'$ , we have*

$$\langle Y', X \rangle - \langle X, Y' \rangle > 0.$$

*Proof.* It is straightforward to check that  $\langle X'_i, X \rangle - \langle X, X'_i \rangle = t_0 n_i > 0$ . Thus assume that  $Y'$  has no direct summand isomorphic to  $X'_i$  for every  $i = 1, \dots, m$ . Consider the commutative diagram (2.1). Since  $\text{Ext}(Y', X/Y') \cong \text{Ext}(Y', X'/Y')$  we have  $\text{Ext}(Y', X/Y') = 0$  by Theorem 13. Moreover, since  $X'$  is polystable, by Lemma 19 we get  $\text{Hom}(X/Y', Y') \cong \text{Hom}(X'/Y', Y') = 0$ . Therefore, we have

$$\begin{aligned}\langle Y', X \rangle - \langle X, Y' \rangle &= \langle Y', X/Y' \rangle - \langle X/Y', Y' \rangle \\ &= \dim \text{Hom}(Y', X/Y') + \dim \text{Ext}(X/Y', Y') - \dim \text{Ext}^2(X/Y', Y').\end{aligned}$$

Define  $X_\cap := \bigcap_{l \in N_q} X_l$ . First assume that  $Y'_0 \cap X_\cap \neq 0$ . For  $l \in N_q$  let  $\tau_l$  be the unique path from  $l$  to  $t(N_q)$  and  $\tau_{t(N_q)}$  be the unique path from  $t(N_q)$  to the root. Let  $T_l$  and  $T_{t(N_q)}$  respectively be the representations such that  $(T_l)_{l'} = k$  for all  $l'$  such that  $l'$  is a tail of some arrow in  $\tau_l$  and  $(T_l)_{l'} = 0$  otherwise. Moreover, we assume  $(T_l)_\rho = \text{id}$  where it makes sense. In the same way, we define  $T_{t(N_q)}$ . Define  $d_0 := \dim Y'_0 \cap X_\cap - \dim Y'_{t(N_q)} \cap X_\cap$ . Then we obtain an exact sequence

$$0 \rightarrow Y' \rightarrow Y'' \rightarrow T_{t(N_q)}^{d_0} \rightarrow 0$$

where we just glue  $T_{t(N_q)}^{d_0}$  to  $Y'_0 \cap X_\cap \neq 0$ . Moreover, if we define  $d_l := \dim Y''_0 \cap X_\cap - \dim Y''_l \cap X_\cap$  for all  $l \in N_q$ , in the same manner we get

$$0 \rightarrow Y'' \rightarrow \tilde{Y} \rightarrow \bigoplus_{l \in N_q} T_l^{d_l} \rightarrow 0.$$

Note that we obviously have  $\tilde{Y} \subset X$ . Now by construction we have  $\dim \text{Ext}^2(S_q, \tilde{Y}) = 0$  and, therefore,  $\dim \text{Ext}^2(X/\tilde{Y}, \tilde{Y}) = 0$ . Note that we have  $\dim \text{Ext}(S_q, \tilde{Y}) = \dim \bigcap_{l \in N_q} \tilde{Y}_l \neq 0$ . Moreover, since the dimension vector is  $\chi$ -positive we have

$$\langle T_l, X \rangle - \langle X, T_l \rangle \leq 0$$

for  $l \in N_q \cup t(N_q)$ . Thus we obtain

$$0 < \langle \tilde{Y}, X \rangle - \langle X, \tilde{Y} \rangle = \langle Y', X \rangle - \langle X, Y' \rangle + \sum_{l \in N_q \cup t(N_q)} (\langle T_l, X \rangle - \langle T_l, Y' \rangle) \leq \langle Y', X \rangle - \langle X, Y' \rangle.$$

Now assume  $Y'_0 \cap X_\cap = 0$ . In particular,  $S_q$  is no direct summand of  $X'/Y'$ . Let  $P(q)$  be the indecomposable projective module corresponding to the vertex  $q$  which is given by the vector spaces  $P(q)_l = k$  for all  $l \succeq q$  and  $P(q)_0 = k$  and the identity map where it makes sense. Now it is straightforward that the injective dimension of  $P(q)$  is one because the cokernel of  $P(q) \hookrightarrow I(0)$  is also injective, where  $I(0)$  denotes the indecomposable injective module corresponding to the root. Since  $\text{Hom}(P(q), Y') = 0$ , there exists a short exact sequence

$$0 \rightarrow P(q)^{\dim(X/Y')_q} \rightarrow X/Y' \rightarrow \overline{X/Y'} \rightarrow 0.$$

Since  $P(q)$  has injective dimension one and  $\dim(\overline{X/Y'})_q = 0$ , we have  $\text{Ext}^2(X/Y', Y') = 0$ . Thus the claim follows.  $\square$

The following lemma is used to prove the next proposition:

**Lemma 22.** *Let a general representation of dimension vector  $\alpha'$  be polystable with canonical decomposition  $\alpha' = \bigoplus_{i=1}^m (\alpha'_i)^{t_i}$  and let  $\beta' \hookrightarrow \alpha'$  such that a general representation of dimension  $\beta'$  has no direct summand of dimension  $\alpha'_i$  for all  $i = 1, \dots, m$ . Then there exist general polystable representations  $X' = \bigoplus_{i=1}^m (X'_i)^{t_i}$  of dimension  $\alpha'$  with  $\underline{\dim} X'_i = \alpha'_i$  of  $N'$  such that there exists a subrepresentation  $Y'$  of  $X'$  of dimension  $\beta'$  satisfying*

$$\dim \text{Ext}(X', Y') \geq \dim \text{Ext}(Y', X').$$

*Proof.* By Theorem 13, we can assume that there exists a subrepresentation  $Y'$  of  $X'$  such that  $\dim \text{Ext}(Y', X'/Y') = 0$ . Since  $\text{Hom}(X', Y') = 0$  and  $\text{Ext}^2(X'/Y', Y') = 0$ , the exact sequence

$$0 \rightarrow Y' \rightarrow X' \rightarrow X'/Y' \rightarrow 0,$$

induces long exact sequences

$$0 \rightarrow \text{Hom}(Y', Y') \rightarrow \text{Ext}(X'/Y', Y') \rightarrow \text{Ext}(X', Y') \rightarrow \text{Ext}(Y', Y') \rightarrow 0$$

and

$$0 \rightarrow \text{Hom}(Y', Y') \rightarrow \text{Hom}(Y', X') \rightarrow \text{Hom}(Y', X'/Y') \rightarrow \text{Ext}(Y', Y') \rightarrow \text{Ext}(Y', X') \rightarrow 0.$$

Since we have  $\dim \text{Ext}(Y', X') \leq \dim \text{Ext}(Y', Y') = -\langle Y', Y' \rangle + \dim \text{Hom}(Y', Y')$ , we get

$$\begin{aligned} \dim \text{Ext}(X', Y') &= \dim \text{Ext}(X'/Y', Y') - \langle Y', Y' \rangle \\ &\geq \dim \text{Ext}(X'/Y', Y') + \dim \text{Ext}(Y', X') - \dim \text{Hom}(Y', Y'). \end{aligned}$$

Since  $\dim \text{Ext}(X'/Y', Y') \geq \dim \text{Hom}(Y', Y')$ , we obtain

$$\dim \text{Ext}(X', Y') \geq \dim \text{Ext}(Y', X').$$

$\square$

Next assume that  $r_0 > 0$  and that no direct summand of  $Y'$  is a direct summand of  $Y$ . Then we have the following:

**Proposition 23.** *Let a general representation with dimension vector  $\alpha'$  be polystable and let*

$$0 \rightarrow X' \rightarrow X \rightarrow S_q^{t_0} \rightarrow 0$$

*be some stable extension of some general representation  $X'$  with  $\underline{\dim} X' = \alpha'$ . Moreover, assume  $\beta' \hookrightarrow \alpha'$ . Let*

$$0 \rightarrow Y' \rightarrow Y \rightarrow S_q^{r_0} \rightarrow 0$$

*with  $1 \leq r_0 \leq t_0$  and  $\underline{\dim} Y' = \beta'$  such that  $Y$  is a subrepresentation of  $X$ . Then we have*

$$\langle Y, X \rangle - \langle X, Y \rangle > 0$$

*and  $Y$  has no direct summand isomorphic to  $S_q$ .*

*Proof.* The second statement is obvious since the first sequence is stable.

By the construction of Proposition 21 we can assume that we generally have  $\text{Ext}^2(S_q, Y') = 0$ . Indeed, otherwise there exists a dimension vector of greater slope such that we generally have  $\text{Ext}(S_q, Y') = 0$ . It follows that  $\text{Ext}^2(S_q, Y) = 0$ , and since we also have  $\text{Ext}(X', Y) = 0$ , we obtain  $\text{Ext}^2(X, Y) = 0$ .

First assume that  $Y' \cong \bigoplus (X'_i)^{r_i}$  with  $r_i \leq t_i$ . Recall that  $\langle Y', X \rangle - \langle X, Y' \rangle = 0$  and consider

$$-\dim \text{Ext}(S_q^{r_0}, X) - \dim \text{Hom}(X, S_q^{r_0}) + \dim \text{Ext}(S_q^{t_0}, Y') = -r_0 \sum_{i=1}^m t_i n_i + t_0 \sum_{i=1}^m n_i r_i.$$

Since the extension is stable, we have that  $t := (t_0, t_1, \dots, t_m)$  is a Schur root of the quiver  $\tilde{Q}$  considered in this section. Moreover, we have  $r := (r_0, r_1, \dots, r_m) \hookrightarrow t$ . In particular, we have  $\langle r, t \rangle - \langle t, r \rangle > 0$ . But, this means

$$-r_0 \sum_{i=1}^m t_i n_i + t_0 \sum_{i=1}^m n_i r_i > 0.$$

Next assume that  $Y' \subsetneq X'$ . In particular, we have  $\text{Ext}(X'/Y', Y') \neq 0$ . Define  $X_\cap := \bigcap_{l \in N_q} X'_l$  and define  $Y_\cap$  analogously. Let  $v := \sum_{i=1}^m n_i t_i$  and  $e_1, \dots, e_v$  be a basis of  $\text{Ext}(S_q, X')$  and let  $e = (e_1, \dots, e_v)$ . Then we consider the representation  $\tilde{X}$  obtained by

$$e : 0 \rightarrow X' \rightarrow \tilde{X} \rightarrow S_q^v \rightarrow 0.$$

Let  $\beta \in \mathbb{N}Q_0$  be a dimension vector. Consider the quiver Grassmannian  $\text{Gr}_\beta(\tilde{X})$  of  $\tilde{X}$  with dimension vector  $\beta$ , i.e.

$$\text{Gr}_\beta(\tilde{X}) = \{N \in \text{Rep}_\beta(Q) \mid N \subset \tilde{X}\}$$

which is a closed (and hence projective) subvariety of  $\prod_{l \in Q_0} \text{Gr}_{\beta_l}(\tilde{X}_l)$ , see for instance [5]. We should mention that quiver Grassmannians are usually defined for quivers without relations. But, since  $\tilde{X}$  satisfies the commutativity relations, it is straightforward to check that every subrepresentation satisfies these relations as well. This is because every linear map associated with  $\tilde{X}$  is injective, all arrows are oriented to the unique root and, moreover, we only have at most one arrow between each two vertices. Thus we actually deal with usual quiver Grassmannians. Consider the natural map  $\Pi_q : \text{Gr}_\beta(\tilde{X}) \hookrightarrow \prod_{l \in Q_0} \text{Gr}_{\beta_l}(\tilde{X}_l) \rightarrow \text{Gr}_{\beta_q}(\tilde{X}_q)$ . In particular,  $\Pi_q$  is a projective morphism and, thus, it is proper and it follows that  $\Pi_q(\text{Gr}_\beta(\tilde{X}))$  is closed in  $\text{Gr}_{\beta_q}(\tilde{X}_q)$ .

Every subrepresentation  $Y'$  of  $X'$  corresponds to a subrepresentation  $\tilde{Y}$  of  $\tilde{X}$  with  $\beta_q := \dim \tilde{Y}_q = \dim Y_\cap$ . Let  $T_{\tilde{Y}}(\text{Gr}_\beta(\tilde{X}))$  be the tangent space at the point corresponding to  $\tilde{Y}$ . First assume that  $\Pi_q$  is surjective. Then we have  $\dim \text{Gr}_\beta(\tilde{X}) \geq \dim \text{Gr}_{\beta_q}(\tilde{X}_q) = \beta_q(v - \beta_q)$ . In particular, we have  $\dim T_{\tilde{Y}}(\text{Gr}_\beta(\tilde{X})) \geq \beta_q(v - \beta_q)$  for all  $\tilde{Y} \in \text{Gr}_\beta(\tilde{X})$ . Moreover, by [5, Proposition 6] we have  $T_{\tilde{Y}}(\text{Gr}_\beta(\tilde{X})) \cong \text{Hom}(\tilde{Y}, \tilde{X}/\tilde{Y})$ . Obviously, we have  $\dim \text{Hom}(Y', X'/Y') \geq \dim \text{Hom}(\tilde{Y}, \tilde{X}/\tilde{Y})$ . Since  $\beta' \hookrightarrow \alpha'$ , applying Theorem 13, we get  $\langle Y', X'/Y' \rangle = \dim \text{Hom}(Y', X'/Y') \geq \beta_q(v - \beta_q) = \dim \text{Ext}(S_q, Y')(v - \dim \text{Ext}(S_q, Y'))$ . Since we have  $r_0 \leq \min\{t_0, \dim \text{Ext}(S_q, Y')\}$ , treating the two cases  $t_0 \leq \dim \text{Ext}(S_q, Y')$  and  $t_0 > \dim \text{Ext}(S_q, Y')$  separately, it is straightforward to check that

$$\dim \text{Hom}(Y', X'/Y') \geq r_0 v - t_0 \dim \text{Ext}(S_q, Y')$$

Now the claim follows because  $\text{Ext}(Y', X'/Y') \neq 0$ .

Next assume that  $\Pi_q$  is not surjective. Since the image of  $\Pi_q$  is closed and since we deal with a stable extension, we generally have that  $\dim \tilde{X}_q \cap Y_\cap = \dim Y_\cap + t_0 - v \geq r_0$ . Thus we have  $\dim \text{Ext}(S_q^{t_0}, Y') \geq t_0(v - t_0 + r_0) \geq r_0 v$  because  $v \geq t_0$ . In summary, applying Lemma 22, we get  $\langle Y, X \rangle - \langle X, Y \rangle > 0$ .

□

Combining Propositions 21 and 23, we obtain the following result:

**Theorem 24.** Let  $\mathcal{N} = \mathcal{N}' \cup \{q\}$  such that  $\mathcal{N}$  and  $\mathcal{N}'$  are related. Let a general representation with dimension vector  $\alpha' \in \mathbb{N}Q(\mathcal{N}')_0$  be polystable with respect to  $\Theta_{\alpha'} = \langle \_, \alpha' \rangle - \langle \alpha', \_ \rangle$  and let

$$0 \rightarrow X' \rightarrow X \rightarrow S_q^{t_0} \rightarrow 0$$

be some stable extension of some general representation  $X'$  with  $\underline{\dim} X' = \alpha'$  such that  $\alpha$  is  $\chi$ -positive where  $\chi$  is given as in Remark 4. Then  $X$  is stable and can be unitarized with the weight  $\chi$ .

We have the following Corollary:

**Corollary 25.** Let  $\mathcal{N}$  and  $\mathcal{N}'$  be related posets such the elements of  $\mathcal{N} \setminus \mathcal{N}'$  are not comparable. Let a general representation of  $Q(\mathcal{N}')$  with dimension vector  $\alpha' \in \mathbb{N}Q(\mathcal{N}')_0$  be polystable with respect to  $\Theta_{\alpha'}$  and let

$$0 \rightarrow X' \rightarrow X \rightarrow \bigoplus_{q \in \mathcal{N} \setminus \mathcal{N}'} S_q^{t_q} \rightarrow 0$$

be an extension of some general representation  $X'$  with  $\underline{\dim} X' = \alpha'$ . Moreover, let the induced extensions  $e_q \in \text{Ext}(S_q^{t_q}, X')$  be polystable such that at least one extension is stable and such that the dimension vector induced by the stable extension is  $\chi$ -positive. Then  $X$  is stable and can be unitarized with some weight  $\chi$ .

*Proof.* We first consider the stable extension

$$0 \rightarrow X' \rightarrow X'' \rightarrow S_q^{t_q} \rightarrow 0.$$

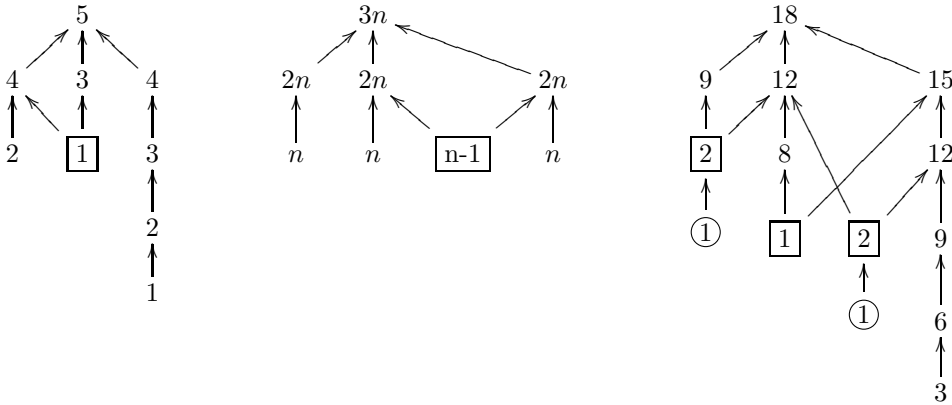
By Theorem 24 we have that  $X''$  can be unitarized with the weight as given in Remark 4. Now we can apply Lemma 9 in order to obtain the result.  $\square$

It is clear that we can apply Lemma 9, after having constructed stable representations using the preceding Corollary, in order to construct stable representations of related posets which do not satisfy the condition of the preceding Corollary.

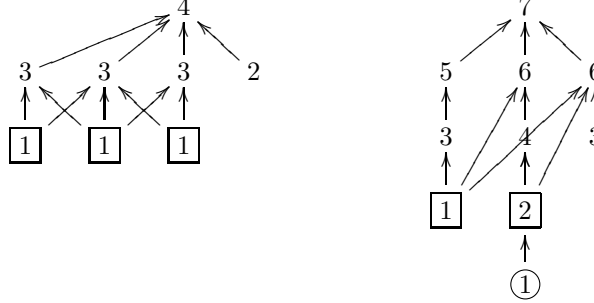
Note that, having constructed a stable representation, by Remark 1, we know a lower bound of the dimension of the moduli space of stable points.

### 3. UNITARIZABLE AND NON-UNITARIZABLE REPRESENTATIONS OF POSETS

**3.1. Some examples of unitarizable representations.** Using the algorithm provided in Section 2.3, for instance in tame cases, fixing a dimension vector one can build families of unitarizable Schurian representations of non-primitive posets that depend on several complex parameters. Below we provide a few examples of such posets and dimension vectors. We start with polystable representations of primitive posets. Then we glue some subspaces using Corollary 25 in order to construct Schurian representations, and afterwards we can glue some extra subspaces as described in Lemma 9.



Here by  $\boxed{i}$  we denote those elements that are glued to the primitive posets as in Corollary 25, and by  $\textcircled{j}$  we denote elements glued to stable representation using Lemma 9. It is clear that one can produce many of such examples. Notice that for some posets and their dimension vectors the provided technique is not applicable, for example in the following cases



In these cases the corresponding representations of the primitive posets are not polystable because the canonical decompositions of the dimension vectors are  $(4; 3; 3; 3; 2) = (3; 2; 2; 2; 2) \oplus (1; 1; 1; 1; 0)$  and  $(7; 3; 5; 4; 6; 3; 6) = (2; 1; 1; 1; 2; 1; 2) \oplus (1; 0; 1; 1; 1; 0; 1) \oplus (4; 2; 3; 2; 3; 2; 3)$ .

**Remark 5.**

- An interesting question is whether the stability condition  $\Theta_\alpha = \langle -, \alpha \rangle - \langle \alpha, - \rangle$  determines a dense subset of Schurian representations of the poset  $\mathcal{N}$  with dimension  $\alpha$  (an analogue of Schofield's Theorem 5). Notice that it is straightforward to check that if  $X$  is an indecomposable quite sincere representation (i.e.  $0 \neq \dim X_q \neq \dim X_0$ ,  $q \in \mathcal{N}$ , and  $\dim X_{q'} < \dim X_q$  if  $q' \prec q$ ) of a poset of representation finite type then  $X$  is stable with the weight  $\Theta_{\dim X}$  (see also [11]).

**3.2. Unitarization of rigid modules.** The *rigidity index* (see for example [14])  $\text{rig}(A_i)$  of a collection  $(A_1, \dots, A_n)$  of matrices  $A_i \in M_m(\mathbb{C})$  is defined by

$$\text{rig}(A_1, \dots, A_n) = m^2(2 - n) + \sum_{i=1}^n \dim(Z(A_i)),$$

where  $Z(X)$  denotes the commutator of the matrix  $X$ , i.e.

$$Z(X) = \{A \in M_m(\mathbb{C}) \mid AX = XA\}.$$

N. Katz, see [14], showed that if  $(A_1, \dots, A_n)$  is an irreducible system of matrices satisfying

$$A_1 \cdot \dots \cdot A_n = I$$

then

$$\text{rig}(A_1, \dots, A_n) \in \{2j \mid j \in \mathbb{Z}, j \leq 1\}.$$

Following Katz we say that the set of matrices is *rigid* if its rigidity index equals 2, otherwise we say that the set of matrices is *non-rigid*.

**Lemma 26.** *Let  $A \in M_m(\mathbb{C})$  be an arbitrary Hermitian matrix with eigenvalues  $\{\lambda_i\}_{i=1}^j$ . Let the multiplicity of each  $\lambda_i$  be  $d_i$ . Then*

$$\dim Z(A) = d_1^2 + \dots + d_j^2.$$

*Proof.* It is clear that the dimension of the commutator of the matrix  $A$  does not depend on the representative of the conjugacy class of  $A$ . Hence we can assume that

$$A = \text{diag}\{\lambda_1, \dots, \lambda_1, \dots, \lambda_j, \dots, \lambda_j\}.$$

Then  $Z(A) = M_{d_1}(\mathbb{C}) \oplus \dots \oplus M_{d_j}(\mathbb{C})$ . Now the statement is obvious.  $\square$

Recall that a module  $X$  is called *rigid* if  $\text{Ext}(X, X) = 0$ .

**Theorem 27.** *Let  $\mathcal{N}$  be a primitive poset of type  $(m_1, \dots, m_n)$ . Assume that  $X$  is an indecomposable rigid strict representation of  $Q(\mathcal{N})$ . Then it is unitarizable with some weight  $\chi$  and the corresponding representation of the algebra  $\mathcal{B}_{\Gamma(\mathcal{N}), \omega}$ , viewed as collection of Hermitian matrices  $A_1, \dots, A_n$ , is rigid.*

*Proof.* By Corollary 12 the unique indecomposable representation  $X$  of a real Schur root can be unitarized. This representation is *rigid* due to  $\dim \text{End}(X) - \dim \text{Ext}(X, X) = 1$ . In this case Schofield's Theorem 5 can be checked easily, see also [12, Lemma 5.1]. We take the Euler characteristic for  $X$ , i.e.

$$\langle X, X \rangle = \dim \text{End}(X) - \dim \text{Ext}(X, X) = \sum_{q \in Q_0} \dim X_q \dim X_q - \sum_{\rho: q \rightarrow q'} \dim X_q \dim X_{q'} = 1.$$

Using Lemma 26 we have that

$$\text{rig}(A_1, \dots, A_n) = m^2(2 - n) + \sum_{i=1}^n \dim(Z(A_i)) = m^2(2 - n) + \sum_{i=1}^n \sum_{j=1}^{m_i+1} d_j^{(i)^2},$$

where  $m = \dim X_0$  and  $d_j^{(i)}$  is the dimension of the  $j$ -th eigenspace of the corresponding Hermitian matrix  $A_i$ , which are given by

$$\begin{aligned} d_1^{(i)} &= \dim X_1^{(i)}, \quad d_j^{(i)} = \dim X_j^{(i)} - \dim X_{j-1}^{(i)}, \quad 2 \leq j \leq m_i, \\ d_{m_i+1}^{(i)} &= \dim X_0 - \dim X_{m_i}^{(i)}. \end{aligned}$$

Then taking  $\langle X, X \rangle$  we get

$$\begin{aligned} \langle X, X \rangle &= (\dim X_0)^2 + \sum_{i=1}^n \sum_{j=1}^{m_i} (\dim X_j^{(i)})^2 \\ &\quad - \sum_{i=1}^n \sum_{j=1}^{m_i-1} (\dim X_j^{(i)})(\dim X_{j+1}^{(i)}) - \sum_{i=1}^n (\dim X_0)(\dim X_{m_i}^{(i)}) \\ &= (\dim X_0)^2 + \frac{1}{2} \sum_{i=1}^n \left( (\dim X_1^{(i)})^2 + \sum_{j=2}^{m_i} ((\dim X_j^{(i)}) - (\dim X_{j-1}^{(i)}))^2 \right) \\ &\quad + \frac{1}{2} \sum_{i=1}^n ((\dim X_0) - (\dim X_{m_i}^{(i)}))^2 - \frac{n}{2} (\dim X_0)^2 \\ &= \frac{1}{2} (m^2(2 - n) + \sum_{i=1}^n \sum_{j=1}^{m_i+1} d_j^{(i)^2}) = \frac{1}{2} \text{rig}(A_1, \dots, A_n). \end{aligned}$$

Since  $\langle X, X \rangle = 1$  because  $X$  is rigid, the corresponding set of the matrices is also rigid.  $\square$

Let  $\mathcal{N}$  be a non-primitive poset, and let  $\mathcal{N}'$  be a related primitive poset, i.e.  $\mathcal{N} = \mathcal{N}' \cup \{q_1, \dots, q_n\}$ . The following Corollary is straightforward:

**Corollary 28.** *Assume that  $X$  is a rigid Schurian representation of  $\mathcal{N}$  such that the following condition holds*

$$\dim X_{q_i} \leq \sum_{l \in N_{q_i}} \dim X_l - (|N_{q_i}| - 1) \dim X_{t(N_{q_i})}$$

*for all  $q_i$  and that the corresponding representation of the related primitive poset is Schurian. Then  $X$  can be unitarized with some weight.*

*Proof.* It is straightforward to see that the corresponding representation  $X'$  of  $\mathcal{N}'$  is rigid. Then we can apply Proposition 27 and Corollary 25 to obtain the statement.  $\square$

### 3.3. ADE classification of unitarizable representations.

**Theorem 29.** *Let  $Q(\mathcal{N})$  be an unbound quiver induced by a poset  $\mathcal{N}$ . Then we have:*

- (1) *Every indecomposable strict representation of  $Q(\mathcal{N})$  is unitarizable if and only if  $Q(\mathcal{N})$  is a Dynkin quiver.*
- (2) *Every Schurian strict representation of  $Q(\mathcal{N})$  is unitarizable if and only if  $Q(\mathcal{N})$  is a subquiver of an extended Dynkin quiver.*
- (3) *There exist families of non-isomorphic unitarizable and non-unitarizable Schurian strict representations which depend on arbitrary many continuous parameters if and only if  $Q(\mathcal{N})$  contains an extended Dynkin quiver as a proper subquiver.*

*Proof.* The first part trivially follows from the previous section observing that in this case all indecomposable representations are Schurian and rigid. Moreover, if the underlying quiver is not of Dynkin type, there always exist non Schurian roots. Indeed, we may consider an isotropic root  $\alpha$ , i.e.  $\langle \alpha, \alpha \rangle = 0$ . Now it is easy to check that  $2\alpha$  is no Schur root, but a root, since the canonical decomposition of  $2\alpha$  is  $\alpha \oplus \alpha$ , see also [26].

*Second part.* The representations that correspond to real Schur roots are obviously rigid and hence unitarizable. In general, by [12, Proposition 5.2] any Schur representation is stable for some linear form  $\Theta$ . Thus following Remark 3 it can be unitarized with some weight. Let us notice that this result (together with the description of possible weights) was alternatively obtained in the series of D.Yakimenko's papers (see [31] and references therein).

*Third part.* Let  $\alpha$  be an indivisible isotropic Schur root of an extended Dynkin quiver. Thus a general representation  $X$  with dimension vector  $\alpha$  is Schurian and can be unitarized by Theorem 11. By adding an extra vertex with fixed dimension  $d$  to a vertex  $q$  with  $\dim X_q > d \geq 1$  to the extended Dynkin quiver we again get a Schurian representation, say with dimension vector  $\tilde{\alpha}$ . In particular,  $\tilde{\alpha}$  is a Schur root. Indeed, we may for instance apply Lemma 7 in order to see that the new representation is a Schurian representation.

It is easy to check that  $\langle \tilde{\alpha}, \tilde{\alpha} \rangle = \langle \alpha, \alpha \rangle + d^2 - d\alpha_q < 0$ . For two general stable representations  $X$  and  $Y$  of dimension  $\tilde{\alpha}$  we have  $\text{Hom}(X, Y) = \text{Hom}(Y, X) = 0$ . Let

$$0 \rightarrow X \rightarrow Z \rightarrow Y \rightarrow 0$$

be a non-splitting exact sequence. Then by Lemma 7, we have  $\text{End}(Z) \subseteq \text{End}(X) = \mathbb{C}$ . Thus,  $Z$  is a semistable Schurian representation which is not stable. Now we can check by a direct calculation that for every weight  $\chi$  we have that  $X$  is a subrepresentation which contradicts  $\chi$ -stability, see Lemma 4.

If we want to glue a vertex  $q$  to some vertex  $q'$  of dimension one we proceed as follows: first we add an extra arrow  $\rho : q' \rightarrow q$  and consider some non-splitting exact sequence  $0 \rightarrow S_q \rightarrow Z \rightarrow X \oplus X' \rightarrow 0$  where  $S_q$  is the simple module corresponding to the vertex  $q$  and  $X$  and  $X'$  are non-isomorphic Schurian of dimension  $\alpha$ , thus  $\text{Hom}(X, X') = 0$ . Then, applying Lemma 7 to the induced sequences  $0 \rightarrow S_q \rightarrow Z' \rightarrow X \rightarrow 0$  and  $0 \rightarrow Z' \rightarrow Z \rightarrow X' \rightarrow 0$  we obtain  $\text{End}(Z) = \mathbb{C}$ . It is easy to check that  $\dim Z$  is an imaginary root which is not isotropic. Now by applying the reflection functor, see [3], corresponding to the vertex  $q$  we again get a Schurian representation  $\tilde{Z}$ . But  $\tilde{Z}$  corresponds to some filtration and we can proceed as in the first case.

The existence of a family of unitarizable representations depending on an arbitrary number of parameters follows in the same manner as Theorem 30.  $\square$

**Remark 6.**

- Let us remark that the first and third part of the theorem hold for posets in general. If the poset is of representation finite type, then each indecomposable representation can be unitarized with some weight (see [11] for the proof). If the poset contains a poset of wild type as a subposet, the same argument as for unbound quivers can be applied. Thus, there is a family of non-isomorphic non-unitarizable



Schurian representations of the poset that depends on arbitrary many continuous parameters.

But it is an open question whether all Schurian representations of tame posets with unoriented cycles are unitarizable. Like in Section 3.1, in many cases it is possible to construct an open subset of unitarizable representations. But as in the case without cycles the constructed weight does not apply for all Schurian representations.

#### 4. COMPLEXITY OF THE DESCRIPTION OF \*-REPRESENTATIONS OF $\mathcal{A}_{\mathcal{N},\chi}$

**Theorem 30.** *Let  $\mathcal{N}$  be a poset of representation wild type. Then it is possible to choose the weight  $\chi_{\mathcal{N}}$  in such a way that for an arbitrary natural number  $n$  there exists a family of non-isomorphic Schurian representations of  $\mathcal{N}$  depending on at least  $n$  complex parameters which can be unitarized with the weight  $\chi_{\mathcal{N}}$ .*

*Proof.* Due to Theorem 1 we only need to consider critical posets of the following types:  $(1, 1, 1, 1, 1)$ ,  $(1, 1, 1, 2)$ ,  $(2, 2, 3)$ ,  $(1, 3, 4)$ ,  $(1, 2, 6)$  and  $(N, 5)$ . Let us consider the dimension vectors  $(2; 1; 1; 1; 1)$ ,  $(4; 2; 2; 2; 1, 2)$ ,  $(6; 2, 4; 2, 4; 1, 2, 4)$ ,  $(8; 4; 2, 4, 6; 1, 2, 4, 6)$  and  $(12; 6; 4, 8; 1, 2, 4, 6, 8, 10)$  respectively of the quivers corresponding to  $(1, 1, 1, 1, 1)$ ,  $(1, 1, 1, 2)$ ,  $(2, 2, 3)$ ,  $(1, 3, 4)$  and  $(1, 2, 6)$  respectively. In order to see that these are Schur roots, we can easily construct a Schurian representation of these dimension vectors. For instance for  $(4; 2; 2; 2; 1, 2)$  we consider a non-splitting short exact sequence  $0 \rightarrow X' \oplus X \rightarrow Y \rightarrow S_5 \rightarrow 0$  where  $X$  and  $X'$  are Schurian representations of dimension vector  $(2; 1; 1; 1; 0, 1)$  with  $\text{Hom}(X, X') = 0$ . As in the proof of Theorem 29 we may apply Lemma 7 to the two induced sequences. The other cases behave analogously.

For the first five dimension vectors  $\alpha_i$  we have that  $\langle \alpha_i, \alpha_i \rangle = -1$ . Hence, following Remark 1 it is possible to choose a two parameter family of Schurian representations. By Theorem 11, a general representation with this dimension vector can be unitarized with the weights

$$\begin{aligned}\chi_{(1,1,1,1,1)} &= (5; 2; 2; 2; 2, 2), \\ \chi_{(1,1,1,2)} &= (8; 4; 4; 4; 2, 3), \\ \chi_{(2,2,3)} &= (12; 4, 4; 4, 4; 2, 3, 4), \\ \chi_{(1,3,4)} &= (16; 8; 4, 4; 4, 2, 3, 4, 4), \\ \chi_{(1,2,6)} &= (24; 12; 8, 8; 2, 3, 4, 4, 4, 4).\end{aligned}$$

For two general unitarizable representations  $X$  and  $X'$  of dimension  $\alpha$  we have  $\text{Hom}(X, X') = \text{Hom}(X', X) = 0$ . The middle term  $Z$  of every non-splitting exact sequence

$$0 \rightarrow X \rightarrow Z \rightarrow X' \rightarrow 0$$

has dimension vector  $2\alpha$ . Moreover, such representations, even if they are not stable, are Schurian by Lemma 7. Thus there exists a non-empty open subset of Schurian representations having the same dimension vector. Following Theorems 5 and 11, there exists a non-empty open subset of representations which can be unitarized with the weight  $2\chi_{\mathcal{N}}$  and, therefore, with the weight  $\chi_{\mathcal{N}}$ , too. The dimension of the corresponding moduli space is  $1 - \langle \dim Z, \dim Z \rangle = 1 - \langle 2\alpha, 2\alpha \rangle = 1 - 4\langle \alpha, \alpha \rangle = 5$ , see Remark 1. Hence there exists a 5-parameters family of non-isomorphic representations having dimension vector  $2\alpha$  which can be unitarized with the same weight  $\chi_{\mathcal{N}}$ . Then iterating the same procedure for the dimension vector  $2\alpha$ , we will obtain the desirable result due to the fact that  $1 - \langle 2^n \alpha, 2^n \alpha \rangle = 1 + 2^{2n}$  grows when iterating.

In the case of the poset  $(N, 5)$  we proceed as follows. We consider the related poset  $(2, 1, 5)$  and the isotropic Schur root  $\alpha = (6; 2, 4; 3; 1, 2, 3, 4, 5)$ . This root is strongly strict. Taking a general polystable representation  $X' = \bigoplus_{i=1}^m (X'_i)^{t_i}$  with  $\underline{\dim} X'_i = \alpha$  and  $X'_i \not\cong X'_j$ , we can consider stable extensions

$$0 \rightarrow X' \rightarrow X \rightarrow S_q^{t_0} \rightarrow 0.$$

where  $(t_0, t_1, \dots, t_m)$  is a Schur root of the dual quiver of the  $m$ -subspace quiver  $S(m)$ . Note that the intersection of the two questioned subspaces is of dimension one. By Theorem 24 any such representation  $X$  is stable, i.e. in particular Schurian, and can be unitarized (for instance with the weight  $\chi_{(N,5)} = (11; 4, 3; 1, 5; 2, 2, 2, 2, 2)$  for  $(t_0, t_1) = (1, 1)$ ). Since  $\alpha$  is an isotropic root we have a one-parameter family of stable representations of dimension  $\alpha$ . In particular, we have a  $d$ -parameter family of polystable representations for every tuple  $(t_1, \dots, t_m)$  with  $\sum_{i=1}^m t_i = d$ .  $\square$

**Corollary 31.** *Let  $\Gamma$  be a star-shaped graph that contains an extended Dynkin graph as a proper subgraph. Then there exists a character  $\omega_\Gamma$  such that the algebra  $\mathcal{B}_{\Gamma, \omega_\Gamma}$  has a family of unitary-nonequivalent irreducible  $*$ -representations which depends on an arbitrary number of continuous parameters.*

*Proof.* Using the previous theorem and the relations between unitarizable systems of subspaces and  $*$ -representations of  $\mathcal{B}_{\Gamma, \omega_\Gamma}$  it is easy to check that letting

$$\begin{aligned}\omega_{(1,1,1,1,1)} &= (5; 2; 2; 2; 2; 2), \\ \omega_{(1,1,1,2)} &= (8; 4; 4; 4; 2, 5), \\ \omega_{(2,2,3)} &= (12; 4, 8; 4, 8; 2, 5, 9), \\ \omega_{(1,3,4)} &= (16; 8; 4, 8, 12; 2, 5, 9, 13), \\ \omega_{(1,2,6)} &= (24; 12; 8, 16; 2, 5, 9, 13, 17);\end{aligned}$$

we obtain the desirable statement.  $\square$

**Remark 7.**

- In [18] it was conjectured that if the graph  $\Gamma$  contains an extended Dynkin graph as a proper subgraph, then there exists a characters  $\omega_\Gamma$  such that the algebra  $\mathcal{B}_{\Gamma, \omega_\Gamma}$  is  $*$ -wild. The previous Corollary gives possible candidates for  $\omega_\Gamma$  among all possible characters, since it is obvious that for such characters the classification task is an extremely difficult problem. In the case when  $\Gamma = (1, 1, 1, 1, 1)$  or  $\Gamma = (1, 1, 1, 2)$  it is known that the algebras  $\mathcal{B}_{\Gamma, \omega_\Gamma}$  are indeed  $*$ -wild with the characters  $\omega_\Gamma$  given as in the previous Corollary. This is due to [23, Section 3.1.3] and private communication with S. Rabanovich. The other cases are unknown by now.

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(Thorsten Weist) FACHBEREICH C - MATHEMATIK, BERGISCHE UNIVERSITÄT WUPPERTAL, D - 42097 WUPPERTAL, GERMANY

*E-mail address:* `weist@math.uni-wuppertal.de`

(Kostyantyn Yusenko) INSTITUTE OF MATHEMATICS NAS OF UKRAINE, TERESCHENKIVSKA STR. 3, 01601 KIEV, UKRAINE

*E-mail address:* `kay.math@gmail.com`